

# Mode regularization of the susy sphaleron and kink: zero modes and discrete gauge symmetry

Alfred Scharff Goldhaber<sup>1</sup>, Andrei Litvintsev<sup>2</sup> and Peter van Nieuwenhuizen<sup>3</sup>  
*C.N.Yang Institute for Theoretical Physics, SUNY at Stony Brook,  
 Stony Brook, NY 11794*

## Abstract

To obtain the one-loop corrections to the mass of a kink by mode regularization, one may take one-half the result for the mass of a widely separated kink-antikink (or sphaleron) system, where the two bosonic zero modes count as two degrees of freedom, but the two fermionic zero modes as only one degree of freedom in the sums over modes. For a single kink, there is one bosonic zero mode degree of freedom, but it is necessary to average over four sets of fermionic boundary conditions in order (i) to preserve the fermionic  $Z_2$  gauge invariance  $\psi \rightarrow -\psi$ , (ii) to satisfy the basic principle of mode regularization that the boundary conditions in the trivial and the kink sector should be the same, (iii) in order that the energy stored at the boundaries cancels and (iv) to avoid obtaining a finite, uniformly distributed energy which would violate cluster decomposition. The average number of fermionic zero-energy degrees of freedom in the presence of the kink is then indeed  $1/2$ . For boundary conditions leading to only one fermionic zero-energy solution, the  $Z_2$  gauge invariance identifies two seemingly distinct ‘vacua’ as the same physical ground state, and the single fermionic zero-energy solution does not correspond to a degree of freedom. Other boundary conditions lead to two spatially separated  $\omega \sim 0$  solutions, corresponding to one (spatially delocalized) degree of freedom. This nonlocality is consistent with the principle of cluster decomposition for correlators of observables.

---

<sup>1</sup>e-mail: goldhab@insti.physics.sunysb.edu

<sup>2</sup>e-mail: litvint@insti.physics.sunysb.edu

<sup>3</sup>e-mail: vannieu@insti.physics.sunysb.edu

# 1 Introduction

The problem of how to compute the one-loop corrections to the mass  $M$  and central charge  $Z$  of supersymmetric (susy) kinks has been the subject of renewed investigations in the past few years. In this article we give a precise prescription for computing the mass of ordinary and susy kinks using mode regularization. The prescription follows from a careful study of bosonic and fermionic zero modes. Differing from previous prescriptions, it yields the accepted result, thus reaffirming mode regularization as a bona fide scheme.

According to standard arguments [1], the mass can be written in terms of differences of sums over zero-point energies, but because the sums themselves are divergent, one must specify how to regularize them. Furthermore, the number and values of the zero-point energies depend on the boundary conditions one imposes on the fields, and hence one must decide which boundary conditions to use, or, more precisely, when and which parts to subtract from the sums over zero-point energies for a given set of boundary conditions. When boundary conditions distort the fluctuations at the boundaries, one must first subtract the extra boundary energy to obtain the mass corrections for the kink and superkink. In addition, as we shall show, certain boundary conditions lead to a uniformly distributed energy density, which also should not be counted as part of the kink mass.

The recent interest in the subject of this paper began with the work [2], where two important questions were posed:

- 1) What is the quantum correction  $M^{(1)}$  to the mass of the kink in supersymmetric  $\lambda\phi^4$  theory?
- 2) If  $M^{(1)}$  is not zero, can the kink remain a BPS-saturated state in the presence of quantum corrections, and if so how does this occur?

The authors of [2] noted that the two regularization methods they used to calculate  $M^{(1)}$ , momentum cutoff and mode cutoff, gave different answers. Moreover, the BPS bound did not seem to be saturated. Since then various regularization schemes have been applied to this problem: momentum cut-off [2, 3, 4], mode cut-off [2, 4], mass-derivative regularization [3, 4], phase-shift methods [5], higher-derivative supersymmetry-preserving regularization [6], dimensional regularization [6] and derivative expansion [7].

From [4] one sees that the inadequacy in the momentum cutoff calculation had to do with the need for smoothing the cutoff so that it becomes well-defined. In the present work we shall show that the mode cutoff calculation was correct, but included a localized boundary energy along with the kink energy.

Nastase, et al. [3] avoided these pitfalls by first evaluating the mass-derivative of the mode sums, which gave better control of the divergences and thus eliminated the need for smoothing the cutoff. Their result for  $M^{(1)}$  agreed with the older work of Schonfeld [8] for the kink-antikink system, suggesting that this indeed is the correct value. They also suggested that there might be an anomaly which would restore the BPS condition.

The MIT group [5] used continuum phase shift methods (avoiding consideration of boundary conditions) to compute the one-loop corrections to the kink energy  $M^{(1)}$  and to the central charge  $Z^{(1)}$ , finding that they are the same, so that the BPS condition is obeyed. They did not ascribe the shift in  $Z$  to an anomaly, instead treating it as a straightforward one-loop

result.

The Minnesota group [6], stimulated by [2] and [3], undertook to attack the second question of [2] directly. There is a beautiful argument originated by Witten and Olive [9]. If the isolated kink has a single ground state, and if perturbative quantum corrections violated the condition  $M = Z$ , then the fact that this would mean the supersymmetry is completely broken implies that there must be a double degeneracy of the ground state, hence a multiplicity discontinuity at  $\hbar = 0$ . In [6] three different methods were used to calculate the one-loop correction to  $Z$ , all methods agreeing that the correction represents a local anomaly, and results in maintaining  $M = Z$ , with  $M$  given by (1). The authors of [6] also computed the one-loop effective superpotential, and extracted from it the effective central charge as the difference of the effective superpotential at  $\pm\infty$ ; this yielded the one-loop correction to  $Z$ .

In [4], the anomaly in the central charge was directly computed using momentum cutoff to regulate both the Dirac delta function appearing in the algebra of the supersymmetry charges and the propagators appearing in the loops; again  $M = Z$  was found.

Perhaps the easiest scheme is mass-derivative regularization [3], according to which one first evaluates the derivative  $\frac{\partial}{\partial m}M = \frac{\partial}{\partial m} \left[ \frac{1}{2} \sum \omega + \Delta M \right]$  of the sums (which is better convergent, so that there is no sensitivity to the form of cutoff) and then integrates w.r.t.  $m$  using the renormalization condition  $M(m=0) = 0$ . Any boundary conditions on the fluctuations for which the divergences in  $M^{(1)}$  cancel are allowed. (There are boundary conditions for which  $M^{(1)}$  diverges, for example susy boundary conditions. The divergence is then a boundary effect which must be subtracted by hand.) A detailed discussion is given in [4]; here we only need the result: the one-loop bosonic, fermionic and supersymmetric corrections to the kink mass are, respectively,

$$M_b^{(1)} = -m\hbar \left( \frac{3}{2\pi} - \frac{\sqrt{3}}{12} \right); \quad M_f^{(1)} = m\hbar \left( \frac{1}{\pi} - \frac{\sqrt{3}}{12} \right); \quad (1)$$

$$M_s^{(1)} = M_b^{(1)} + M_f^{(1)} = -\frac{m\hbar}{2\pi}$$

These values are now accepted by all workers in the field [3-7].

The problem to be solved is thus how to obtain these results with the other regularization schemes and other boundary conditions. The most commonly used schemes are energy cut-off (= momentum cut-off) and mode regularization. Although each has been in use for decades, we claim that each needs modifications. For energy cut-off regularization (in which one first computes each of the sums up to the same given energy  $\Lambda$ , and then takes the limit  $\Lambda \rightarrow \infty$ ) it was found that a simple modification makes the sum well defined, and reproduces (1): instead of an abrupt cut-off at  $\Lambda$  one needs a smooth cut-off which interpolates between the zero-point energies in the topological and trivial sectors [4].

In this article we repair mode number regularization. The basic idea of this scheme is to subtract an equal number  $N$  of modes  $\omega_n^{(0)}$  in the trivial sector from the modes  $\omega_n$  in the topological sector, and then to take the limit  $N \rightarrow \infty$ , but a problem arises whether one should include some or all or none of the zero modes in this counting. This issue has turned out to be surprisingly complicated, and for pedagogical reasons we shall first deduce in section 2 the correct rules by requiring that they reproduce the result in (1).

The problem then obviously is to justify these rules. We shall first consider a kink-antikink configuration, which lies in the trivial sector (having no overall winding number) so that standard manipulations of quantum field theory are still reliable. For such a system the energy located at the boundaries in the kink-antikink sector cancels the same quantity in the trivial sector if one uses the same boundary conditions in both sectors<sup>4</sup>, so the mass of the kink is then just one-half of the sums over the zero-point energies plus the counter term for mass renormalization (the latter will be given in section 2). (For periodic or antiperiodic boundary conditions in the kink-antikink system, there is not even any localized boundary energy because these boundary conditions are translationally invariant). For the bosonic case one finds for the kink mass (putting  $\hbar = 1$ )

$$M_b^{(1)} = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^N \left( \frac{1}{2} \omega_n^b - \frac{1}{2} \omega_n^{b,(0)} \right) + \Delta M_b \quad (2)$$

where  $\frac{1}{2} \omega_n^b$  are the zero point energies for the bosonic fluctuations around the kink-antikink background,  $\frac{1}{2} \omega_n^{b,(0)}$  those around the trivial background, and  $\Delta M_b$  is the counterterm for a single kink.

Next we consider the susy kink. Except for a unique value of the strength of the Yukawa coupling of the fermions, the susy of the action is explicitly and completely broken, but in the  $\bar{K}K$  (antikink-kink) background zero modes remain. We shall therefore generalize our approach and consider arbitrary kinks with fermions, and not only susy kinks. The action we use contains a Yukawa term  $-c\sqrt{\frac{\lambda}{2}}\phi\bar{\psi}\psi$  with  $\psi$  a real two-component spinor and  $c = 1$  for susy. The correction to the mass of the kink is given by

$$M_s^{(1)} = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^N \left[ \frac{1}{2} \omega_n^b - \frac{1}{2} \omega_n^{b,(0)} - \frac{1}{2} \omega_n^f + \frac{1}{2} \omega_n^{f,(0)} \right] + \Delta M_{[b+f]} \quad (3)$$

where  $b$  denotes bosonic frequencies and  $f$  fermionic ones and the counterterm  $\Delta M_{[b+f]} = \Delta M_b + \Delta M_f$  is due to both bosonic and fermionic loops.

There are various ways to describe a kink-antikink background. By far the most used is the configuration which describes a kink centered at  $L/2$  for  $x \geq 0$  and an antikink centered at  $-L/2$  for  $x < 0$ .

---

<sup>4</sup> If one first considers a finite number of modes in the sector where the classical scalar field is constant, and then slowly turns on the kink-antikink configuration by pulling the scalar field around  $x = 0$  away from its constant value, the mode energies move from their values in the trivial background to their values in the  $\bar{K}K$  configuration. This is the justification for mode regularization, as in (2). Since a change in the background away from the boundaries will not change the (localized or delocalized) boundary energy, the latter (if present) cancels between the  $\bar{K}K$  case and the trivial case.

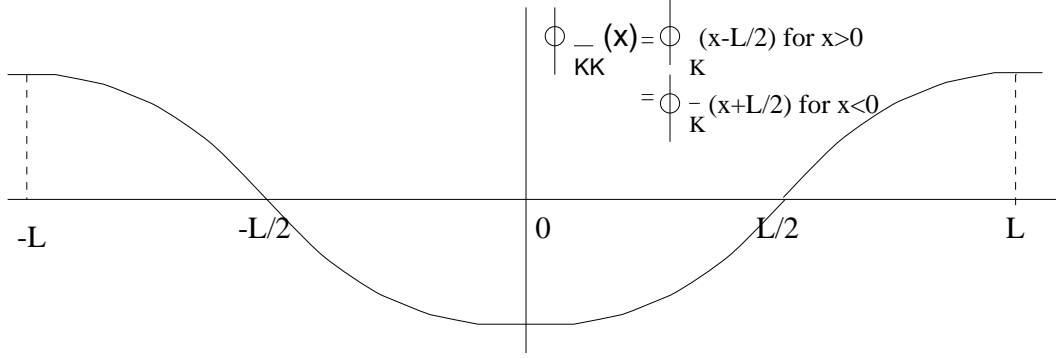


Figure 1. *The kink-antikink configuration.*

This configuration has the slight drawback that at  $x = 0$  the field  $\phi(x)$  is not differentiable (the left- and right- derivatives differ), so that  $\phi(x)$  is not a solution of the field equation. We shall call this configuration  $\phi_{\bar{K}K}(x)$ . Another configuration one might consider is everywhere differentiable, but nowhere a solution of the field equations:  $\phi_{K+\bar{K}}(x) = \phi_K(x) + \phi_{\bar{K}}(x) + \frac{\mu}{\sqrt{\lambda}}$ . For large positive  $x$  the sum of  $\phi_{\bar{K}}(x)$  and  $\frac{\mu}{\sqrt{\lambda}}$  vanishes and one obtains the usual kink solution  $\phi_K(x)$ , while for large negative  $x$  one is left with  $\phi_{\bar{K}}(x)$ . However, it is difficult to determine the spectrum of fluctuations around this background, and we shall not use it below. A third configuration one might consider is one of the sphaleron solutions  $\phi_{sph}(x)$  of [10], which are defined on the interval  $-L \leq x \leq L$  with the periodic boundary conditions  $\phi_{sph}(-L) = \phi_{sph}(L)$  and  $\phi'_{sph}(-L) = \phi'_{sph}(L)$ . (The sphaleron is thus defined on a circle). For our purposes the sphaleron solution which becomes one kink-antikink pair as  $L$  tends to infinity is the relevant one, and we shall hereafter refer to it as ‘the sphaleron’. We shall see that our results for mode regularization are the same for the  $\bar{K}K$  background as for the sphaleron background.

Our main conclusion for the bosonic kink-antikink system, to be derived below, is that both translational zero modes should be taken into account in the sum over zero-point energies (2). When the kink and antikink are not infinitely far apart and are described by the sphaleron background, one still has a zero mode with  $\omega = 0$  for translations, while a second mode has  $\omega^2 < 0$  and indicates an instability (the kink is attracted by the antikink). For  $\omega = 0$ , one has the usual collective coordinate  $\hat{X}$  (the Hamiltonian does not depend on  $\hat{X}$ ) for translations and its canonically conjugate momentum  $\hat{P}$ , so that  $\hat{X}$  and  $\hat{P}$  form a canonical pair and correspond to one term in the sum over  $\omega_n^b$  in (2). For  $\omega^2 < 0$  the solutions with  $e^{\pm|\omega|t}$  define one pair of canonical variables and thus another term in the sum over zero-point energies. (The Hamiltonian depends in this case on both canonical variables). In the  $\phi_{\bar{K}K}$  background there is one pair of solutions with  $\omega^2 < 0$ , but the other pair of solutions now has  $\omega^2 > 0$ . As usual the solutions with  $e^{\pm i\omega t}$  then describe one degree of freedom. Thus there are still two degrees of freedom associated with the bosonic zero modes.

For fermions the situation is quite different. For the sphaleron background we find two fermionic solutions with  $\omega^2 = 0$  (see (18)). The corresponding operators  $\beta$  and  $\gamma$  satisfy  $\beta^2 = \gamma^2 = 1$  and  $\{\beta, \gamma\} = 0$  and thus again there is one degree of freedom. Next we study the  $\phi_{\bar{K}K}$  system with a finite separation  $L$ . Then we find two solutions of the Dirac equation

with the same (very small)  $\omega^2 > 0$  which are normalizable and which enter in the second quantization of the fermionic fluctuation field  $\psi(x, t)$  (for  $x > 0$ ) as follows

$$\psi(x, t) = b \left( \begin{array}{c} \psi_K(x) \\ \frac{i}{\omega}(\partial_x + c\sqrt{2\lambda}\phi_K)\psi_K(x) \end{array} \right) e^{-i\omega t} + b^\dagger \left( \begin{array}{c} \psi_K(x) \\ -\frac{i}{\omega}(\partial_x + c\sqrt{2\lambda}\phi_K)\psi_K(x) \end{array} \right) e^{i\omega t} \quad (4)$$

Here  $\psi_K(x)$  is a real normalizable function,  $\phi_K(x)$  is the kink solution and  $c = 1$  for the susy case. Clearly  $b$  and  $b^\dagger$  form **one** conjugate pair, hence one degree of freedom. Therefore the final effect of fermionic zero modes amounts to one term in the sum over  $\omega_n^f$  in (3).

Moving the kink and antikink apart, one obtains a free kink and a free antikink, each having its own zero mode in the Dirac equation (even for  $c \neq 1$ ). The problem then is how to perform mode number regularization for a single isolated kink. As we shall discuss later, for a canonical description one must take into account four sets of boundary conditions, and average the results. Then the fermionic “half degree of freedom” of the kink appears as a change in degrees of freedom (from vacuum) by unity in one pair of boundary conditions, and no change at all in the other pair.

Closer inspection reveals that for certain boundary conditions there is exactly one fermionic zero-energy solution. As the ground state then is an eigenstate of the operator which appears as the coefficient of this solution in the fermion field, the Fock space for this system is half as big as one might have expected, meaning that by this elementary criterion there is no zero-energy fermion degree of freedom.

In string theory one encounters a similar situation with respect to the zero mode of the coordinate ghost (denoted by  $c_0$ ). In that case BRST cohomology shows that states with  $c_0$  are BRST exact, so that there is no doubling of the number of states. In our case we do not have BRST symmetry to remove half of the states of the Fock space, but we shall show that one can divide Fock space into two sectors, such that all operators map states of one sector into states of the same sector. The other sector is then a  $Z_2$ -gauge copy of the first, and as a discrete  $Z_2$ -gauge symmetry in string theory can be promoted to a continuous symmetry, a BRST approach may be possible also for the susy kink.

## 2 The correct rules for mode regularization

In this section we demonstrate that for the kink-antikink system counting two zero modes in the bosonic spectrum but only one in the fermionic spectrum gives the correct answers (1) for both the susy and the bosonic case. In the next sections we derive these rules. After a brief review of the properties of the spectra, the actual calculation of the mass is performed in (15) and (16).

The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}U^2(\phi) - \frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}c\frac{dU}{d\phi}\bar{\psi}\psi \quad (5)$$

where for the kink  $U[\phi(x)] = \sqrt{\frac{\lambda}{2}}\left(\phi^2 - \frac{\mu_0^2}{\lambda}\right)$  and  $\bar{\psi} = \psi^\dagger i\gamma^0$ . We shall use the representation  $\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and take  $\psi$  real. This system has susy when  $c = 1$ ,

and then the susy transformation rules are  $\delta\phi = \bar{\epsilon}\psi$  and  $\delta\psi = \gamma^\mu\partial_\mu\phi\epsilon - U\epsilon$ . The theory with or without fermions is renormalized by replacing  $\mu_0^2$  by  $\mu^2 + \delta\mu^2$ , where for the susy case  $(\delta\mu^2)_s = \frac{\lambda\hbar}{4\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{\sqrt{k^2+m^2}}$  (we put  $m^2 = 2\mu^2$ ). For the bosonic case  $(\delta\mu^2)_b = \frac{3\lambda\hbar}{4\pi} \int_{-\Lambda}^{\Lambda} \frac{dk}{\sqrt{k^2+m^2}}$ . The one-loop correction to the mass of the supersymmetric kink is given by (3) with counter-term  $\Delta M_{[b+f]} = \frac{m}{\lambda}(\delta\mu^2)_s$ . For the bosonic case we use (2) with  $\Delta M_b = \frac{m}{\lambda}(\delta\mu^2)_b$ . The value of  $\delta\mu^2$  follows from requiring absence of tadpoles and the value of  $\Delta M$  follows from replacing  $\phi$  by  $\phi_K$  in  $\int_{-\infty}^{\infty} \frac{1}{2}U^2(\phi)dx$  and retaining the term linear in  $\delta\mu^2$ . For details see [2].

Expanding the bosonic field around the background configuration  $\phi(x, t) = \phi_{\bar{K}K}(x, t) + \eta(x)e^{-i\omega t}$  one finds the following equation for the bosonic modes  $\eta$ :

$$\omega^2\eta + \partial_x^2\eta - [(U')^2 + U''U]\eta = 0 \quad (6)$$

The solutions of this equation for the single kink background can be found explicitly (see, for example, [1]). The spectrum consists of a translational zero mode, a bound state  $\omega_B = \sqrt{3}m/2$ , and a continuum of states. From this information one can easily extract the spectrum of the kink-antikink configuration at large  $L$ .

For the fermionic fields we set  $\psi(x, t) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} e^{-i\omega t}$  and obtain the Dirac equation

$$\begin{aligned} i\omega\psi_2 + (\partial_x + U')\psi_1 &= 0 \\ -i\omega\psi_1 + (-\partial_x + U')\psi_2 &= 0 \quad . \end{aligned} \quad (7)$$

For  $\omega = 0$ , the single kink (or antikink) in a box defined by  $-L/2 \leq x \leq L/2$  has two fermionic solutions with periodic boundary conditions, namely the zero mode attached to the kink, obeying

$$\partial_x\psi_1 + U'\psi_1 = 0 \quad , \quad (8)$$

where  $U' = m \tanh \frac{mx}{2}$ , and a second zero mode attached to the boundary, obeying

$$\partial_x\psi_2 - U'\psi_2 = 0 \quad . \quad (9)$$

The solutions to these equations are

$$\psi_I = \begin{pmatrix} a_1/\cosh^2 \frac{mx}{2} \\ 0 \end{pmatrix} \quad , \quad \psi_{II} = \begin{pmatrix} 0 \\ a_2 \cosh^2 \frac{mx}{2} \end{pmatrix} . \quad (10)$$

(for  $c \neq 1$  the power 2 becomes  $2c$ ).

For  $\omega \neq 0$  one may express each component of  $\psi$  in terms of the other, in which case (7) becomes

$$\begin{aligned} \omega^2\psi_1 + \partial_x^2\psi_1 - (U')^2\psi_1 + U''\partial_x\phi\psi_1 &= 0 \\ \omega^2\psi_2 + \partial_x^2\psi_2 - (U')^2\psi_2 - U''\partial_x\phi\psi_2 &= 0 \quad . \end{aligned} \quad (11)$$

For the case of a kink one may use the Bogomol'nyi equation  $\partial_x\phi = -U$ , and (11) becomes

$$\begin{aligned} \omega^2\psi_1 + \partial_x^2\psi_1 - [(U')^2 + U''U]\psi_1 &= 0 \\ \omega^2\psi_2 + \partial_x^2\psi_2 - [(U')^2 - U''U]\psi_2 &= 0 \quad . \end{aligned} \quad (12)$$

These are Schrödinger-type equations, and the first of them is the same as (6). For the antikink sector the Bogomol'nyi equation reads  $\partial_x \phi = U$ , so one must exchange  $\psi_1$  and  $\psi_2$  in (12) keeping  $U$  unchanged.

In general, the equations (6) and (12) for a kink-antikink system on  $-L \leq x \leq L$  can be written as

$$\omega^2 f(x) + \partial_x^2 f(x) - V(x)f(x) = 0 \quad (13)$$

The potentials  $V(x)$  for bosonic and fermionic fluctuations are sketched in Fig. 2.

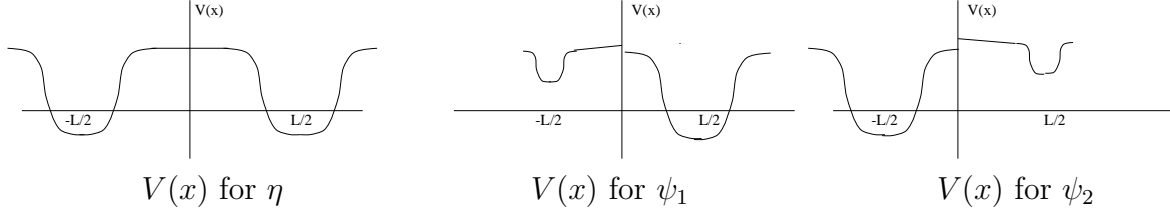


Figure 2. Potentials for the bosonic and fermionic fluctuations in the kink-antikink system.

A plane wave incident from the right acquires a phase shift  $\delta(k)$  in the deeper potential  $V(x) = \frac{1}{2}m^2(3 \tanh^2 \frac{mx}{2} - 1)$  and a phase shift  $\delta(k) + \theta(k)$  in the shallower potential  $V(x) = \frac{1}{2}m^2(\tanh^2 \frac{mx}{2} + 1)$ . Thus on the far right  $\psi_1 \sim e^{i[kx + \delta(k) + \frac{1}{2}\theta(k)]}$ , on the far left  $\psi_1 \sim e^{i[kx - \delta(k) - \frac{1}{2}\theta(k)]}$  while near  $x = 0$  one has  $\psi_1 \sim e^{i[kx + \frac{1}{2}\theta(k)]}$ . The phase shifts are given in Fig. 3

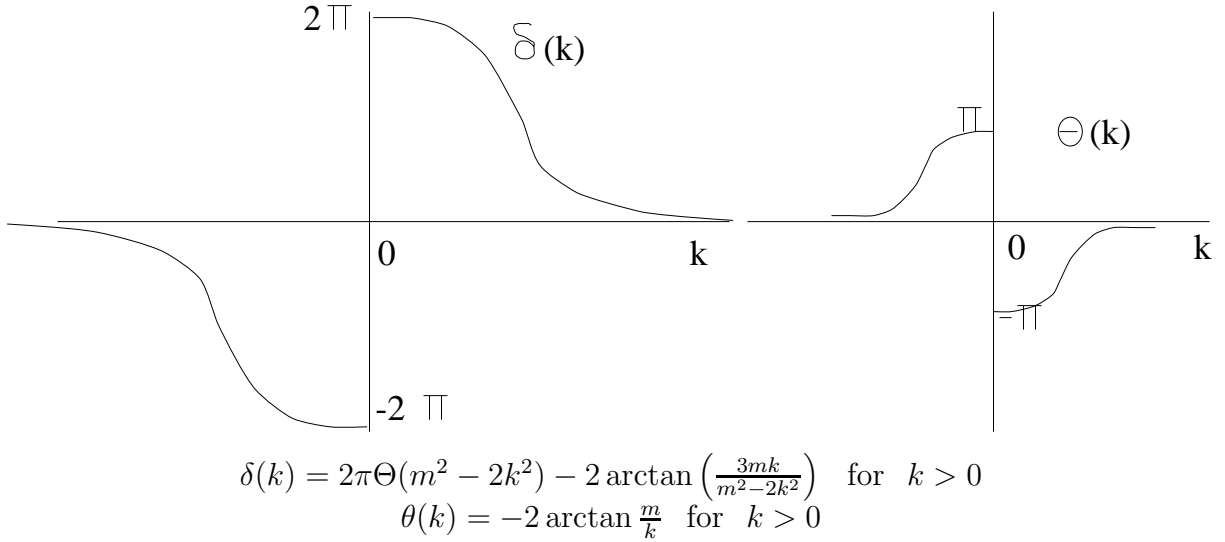


Figure 3. The phase shifts for the bosonic and fermionic fluctuations.

The phase shifts are defined up to  $2\pi$ . Different authors use different expressions for these phase shifts, which nevertheless lead to the same answers.



For definiteness we choose a particular set of boundary conditions although for the kink-antikink system to which we now turn, the results do not depend on which choice one makes. We choose the boundary conditions of [8],  $\psi_1(L) = \psi_1(-L) = \eta(L) = \eta(-L) = 0$  where  $\eta(x)$  are the fluctuations of the bosonic field around the background (the background is constant in the trivial sector, and equal to the kink-antikink combination of Fig. 1 in the nontrivial sector). These boundary conditions are susy if one considers a kink background but they are not susy for the kink-antikink background, because the kink background breaks half of the susy while the antikink background breaks the other half. The boundary conditions lead to the following quantization rules

$$\begin{aligned} k_n^b L + \delta(k_n^b) &= \frac{\pi n}{2} \quad \text{for bosons} \\ k_m^f L + \delta(k_m^f) + \frac{1}{2}\theta(k_m^f) &= \frac{\pi m}{2} \quad \text{for fermions} \end{aligned} \quad (14)$$

where the integers  $n$  and  $m$  are non-negative. Solutions of equations (14) only exist for  $n \geq 4$  and  $m \geq 3$ . Clearly  $k_n^b = k_n^{b,0} - \delta(k_n^{b,0})/L + \mathcal{O}(1/L^2)$ , with a similar expression for  $k_m^f$ .

For large  $k$ , the bosonic and fermionic levels with the same  $m$  and  $n$  approach each other, but the bosonic energy is always a bit smaller than the fermionic energy. For any given  $n = m$  there is still a small interval of momenta such that if we would pick a cut-off in this interval, then the corresponding bosonic level will be below the cut-off, and the corresponding fermionic level above the cut-off. In ref. [8] Schonfeld computes (3) and he implicitly excludes this possibility as non-generic, having very small probability in comparison with the chance to include both corresponding levels (see below his eq. (2.44)). With this prescription, Schonfeld's procedure becomes in effect a mode cut-off method, with one more fermionic than bosonic mode in the continuous spectrum.

We are now ready to apply our counting rules. Let us start with the bosonic case. We include two zero modes in our counting. The bosonic mass correction for the kink-antikink system is given by (reinserting  $\hbar$ )

$$\begin{aligned} 2M_b^{(1)} &= \left( 2 \times 0 + 2\frac{\hbar}{2}\omega_B + \frac{\hbar}{2} \sum_{n=4}^N \sqrt{(k_n^b)^2 + m^2} \right) - \frac{1}{2} \sum_{n=0}^N \hbar \sqrt{(k_n^0)^2 + m^2} + 2\Delta M_b \\ &= \hbar\omega_B - 2m\hbar + \frac{1}{2}\hbar \sum_{n=4}^N \left( \sqrt{(k_n^b)^2 + m^2} - \sqrt{(k_n^0)^2 + m^2} \right) + 2\Delta M_b \\ &= \hbar\omega_B - 2m\hbar + \frac{1}{2}\hbar \int_0^\Lambda \frac{dk}{(\pi/2)} \frac{d\omega}{dk} (-\delta(k)) + 2\Delta M_b \\ &= \hbar\omega_B - 2m\hbar - \hbar \frac{\omega\delta(k)}{\pi} \Big|_0^\Lambda + \left[ \hbar \int_0^\Lambda \frac{dk}{\pi} \omega\delta'(k) + 2\Delta M_b \right] = \hbar\omega_B - \frac{3m\hbar}{\pi} + 2 \left[ -\frac{\sqrt{3}m\hbar}{6} \right] \end{aligned} \quad (15)$$

We used  $\omega\delta(k) = 2\pi m$  at  $k = 0$  and  $\omega\delta(k) = 3m$  for  $k \rightarrow \infty$ , while  $\hbar \int_0^\Lambda \frac{dk}{2\pi} \omega\delta'(k) + \Delta M_b = -\frac{\hbar\sqrt{3}m}{6}$  (see [2], eq (14)). With  $\omega_B = \frac{\sqrt{3}}{2}m$  we indeed get (1). Note that if we would forget about the unstable mode, i.e. if we would take only one zero mode into account, the result for  $M_{bos}^{(1)}$  would be divergent. Also omitting to include any zero mode in (2) yields a divergent answer.

Next consider the supersymmetric case. The bosonic and fermionic contributions in the trivial sector cancel. We know already that we need two bosonic zero modes. We claim that one should take only one fermionic zero mode into account (for reasons to be explained later). The mass correction of the susy kink-antikink system is then given by

$$\begin{aligned}
2M_s^{(1)} &= \left[ 2 \times 0 + 2\frac{1}{2}\hbar\omega_B + \frac{1}{2} \sum_{n=4}^N \hbar\sqrt{(k_n^b)^2 + m^2} \right] \\
&\quad - \left[ 0 + 2\frac{1}{2}\hbar\omega_B + \frac{1}{2} \sum_{n=3}^N \hbar\sqrt{(k_n^f)^2 + m^2} \right] + 2\Delta M_{[b+f]} \\
&= -\frac{1}{2}m\hbar + \frac{1}{2}\hbar \sum_{n=4}^N \left( \sqrt{(k_n^b)^2 + m^2} - \sqrt{(k_n^f)^2 + m^2} \right) + 2\Delta M_{[b+f]} \\
&= -\frac{1}{2}m\hbar + \frac{1}{2}\hbar \int_0^\Lambda \frac{dk}{(\pi/2)} \left( \frac{d}{dk} \sqrt{k^2 + m^2} \right) \left( \frac{1}{2}\theta(k) \right) + 2\Delta M_{[b+f]} \\
&= -\frac{1}{2}m\hbar + \frac{\hbar}{2\pi} \omega\theta(k)|_0^\infty = -\frac{1}{2}m\hbar - \frac{m\hbar}{\pi} + \frac{1}{2}m\hbar = -\frac{m\hbar}{\pi} \tag{16}
\end{aligned}$$

We used  $k_n^b - k_n^f = \frac{1}{2L}\theta(k_n) + \mathcal{O}(1/L^2)$ , and  $-\int_0^\Lambda \frac{dk}{4\pi} \sqrt{k^2 + m^2} \frac{d}{dk}\theta(k) + \Delta M_{[b+f]} = 0$  (see [2], eq (59)). The expression in (16) again gives the accepted result (1). Note that if one assumed either 0 or 2 fermionic zero modes, the answer would be infinite.

In [4] the corresponding analysis for periodic boundary conditions was found to yield exactly the same result and we shall use those conditions in what follows.

### 3 Supersymmetric sphalerons on a circle

We turn now to a justification of the rules that one should take two bosonic zero modes and one fermionic zero mode into account when using mode regularization for  $M_s^{(1)}$ .

Consider the first sphaleron solution on a circle with large circumference  $2L$ ; it describes a kink-antikink system with periodic boundary conditions [10]. The bosonic fluctuations around this background have been analyzed in [10], and we quote the result. For  $L = \infty$  there are two zero modes, but when  $L$  is reduced (bringing the kink and the antikink together) one of the zero modes becomes unstable ( $\omega^2 < 0$ ), while rotational invariance guarantees that the other zero mode remains at zero ( $\omega^2 = 0$ ). The value for the unstable mode for large  $L$  (adapted to our normalization) is  $\omega^2 = -48m^2 \exp(-mL)$ .

We now extend the sphaleron solution to the case with fermions present. We need the background solution. It is given by

$$\phi_n(x) = \frac{mkb}{\sqrt{2\lambda}} \text{sn}(bx, k) \tag{17}$$

where  $b = [1/(2(1+k^2))]^{1/2}m$  and  $\text{sn}(bx, k)$  is an elliptic function [11]. All we need is that this function satisfies the classical field equations, is odd in  $x$ , where  $x = 0$  is the center of the kink (or antikink, of course), and smooth on the circle.

We can now settle the issue of the fate of the fermionic zero mode in one line: the Dirac Hamiltonian due to (7) is manifestly self-adjoint, so that  $\omega$  is real and thus  $\omega^2$  is nonnegative. Thus all one has to study is whether there are any zero modes, and how many. For  $\omega = 0$  the Dirac equation in (8) and (9) has as solutions

$$\psi_I = \beta \begin{pmatrix} a_1 \exp[-2bk \int \text{sn}(bx, k) dx] \\ 0 \end{pmatrix}, \quad \psi_{II} = \gamma \begin{pmatrix} 0 \\ a_1 \exp[2bk \int \text{sn}(bx, k) dx] \end{pmatrix}. \quad (18)$$

Because the function  $\text{sn}(bx, k)$  is odd in  $x$ , the spinors at opposite points from the center of the kink are equal, and hence if one goes around the circle from  $x = 0$  in either direction, one reaches the same value for the spinor at the antipodal point. Thus the solutions are continuous. They are actually smooth (differentiable) because the Dirac equation is first order in derivatives.

Our conclusion is that for a sphaleron background the fermionic spectrum has two zero modes, the same as for an infinitely separated kink and antikink. The canonical equal-time anticommutation relations read according to the Dirac formalism  $\{\psi_i(x, t), \psi_j(y, t)\} = \delta_{ij}\delta(x - y)$ . It follows that the operators  $\beta$  and  $\gamma$  satisfy  $\beta^2 = \gamma^2 = 1$  and  $\{\beta, \gamma\} = 0$ : hence  $\beta + i\gamma$  and  $\beta - i\gamma$  form the annihilation and creation operators for one degree of freedom.

It is possible to give a physical explanation why for fermions  $\omega^2 > 0$  for the  $\bar{K}K$  system but  $\omega^2 = 0$  for the sphaleron system. Consider a configuration  $\phi(x)$  for the boson field on the circle vanishing at two points  $x = x_1$  and  $x = x_2$  which are not antipodal, such that  $\phi(x)$  is a solution of the field equation except at  $x = x_1$  and  $x = x_2$ . (Such a solution exists because it describes according to the usual mechanical analogue a ball oscillating around the bottom of the inverted potential. For the segment where  $x_1$  and  $x_2$  are nearer to each other, the value of  $\frac{\partial}{\partial x}\phi(x)$  at  $x_1$  and  $x_2$  is smaller than for the other segment where  $x_1$  and  $x_2$  are further apart.) By making the radius of the circle large enough, the effect of the kink and antikink on the fermions can be neglected. (The kink and antikink fields only differ from their asymptotic values over a distance  $\Delta x \sim \mu^{-1}$ ). Note that the zero mode of the kink and antikink increases exponentially on one segment while it decreases exponentially on the other segment. It becomes clear that a spinor which is transported along the circle can not be periodic because the segments have different length.

Thus in the sphaleron background the assumption  $\omega = 0$  can not be satisfied. For large  $L$  but fixed  $x_1 - x_2$  this looks like the  $\bar{K}K$  solution on the infinite line except that now the discontinuities in  $\phi'(x)$  arise at the centers of the kink and antikink, instead of in between.

## 4 Zero modes of the $\phi_{\bar{K}K}$ system

We now study the discrete spectrum of the kink-antikink system on an infinite line with the background of Fig. 1.

The bosonic modes are the solutions of (6). The solutions for  $x > 0$  are given by [1]

$$\eta_K(\omega, x) = \exp ik(x - \frac{L}{2}) \left[ -3 \tanh^2 \frac{m(x - \frac{L}{2})}{2} + 1 + \frac{4k^2}{m^2} + \frac{6ik}{m} \tanh \frac{m(x - \frac{L}{2})}{2} \right] \quad (19)$$

with  $\omega^2 = k^2 + m^2$ . For the solutions with  $\omega^2 < m^2$  on  $0 \leq x < \infty$  which are square-integrable, we take  $k = i\kappa$  with  $\kappa > 0$ . The solution of (6) should also have the same left-derivative as right-derivative at  $x = 0$  because the potential is continuous.

When one considers the zero modes of the kink and the antikink together, taking the symmetric combination, the resulting function is still continuous at  $x = 0$ , but the derivative is discontinuous (there is a cusp). Making  $\omega^2 < 0$  decreases the curvature of the solution, and one can find a value of  $\omega^2$  such that also the derivative becomes continuous. Hence, the lowest mode in the  $\bar{K}K$  system is symmetric and has negative  $\omega^2$ . Using (19) one finds  $\omega^2 = -12m^2 \exp(-mL)$  for large  $L$ . The next mode is antisymmetric. One can find the value of  $\omega^2$  for this solution by requiring that  $\psi_K(\omega, x) = 0$  at  $x = 0$ . There is no further condition involving derivatives, because if we take the antisymmetric combination of the solutions which vanishes at  $x = 0$ , its derivative is there continuous. We find that this second mode has  $\omega^2 > 0$ , namely  $\omega^2 = 12m^2 \exp(-mL)$  for large  $L$ . Note that contrary to the sphaleron there is now no longer a zero mode. The reason is clear: the zero mode would be the derivative of the configuration  $\phi_{\bar{K}K}(x)$  in Fig. 1, but this derivative is discontinuous at  $x = 0$ , and therefore not a solution.

Let us now turn to the fermionic sector. The fermionic modes which should become zero modes as  $L$  tends to infinity are given by

$$\psi(x, t) = \alpha \begin{pmatrix} \psi_K^R(\omega, x) \\ \frac{i}{\omega}(\partial_x + c\sqrt{2\lambda}\phi_K(x))\psi_K^R(\omega, x) \end{pmatrix} e^{-i\omega t} + \beta \begin{pmatrix} \psi_K^R(\omega, x) \\ -\frac{i}{\omega}(\partial_x + c\sqrt{2\lambda}\phi_K(x))\psi_K^R(\omega, x) \end{pmatrix} e^{i\omega t} \quad (20)$$

for positive  $x$ , and

$$\psi(x, t) = \gamma \begin{pmatrix} \frac{i}{\omega}(\partial_x - c\sqrt{2\lambda}\phi_{\bar{K}}(x))\psi_K^L(\omega, x) \\ \psi_K^L(\omega, x) \end{pmatrix} e^{-i\omega t} + \delta \begin{pmatrix} -\frac{i}{\omega}(\partial_x - c\sqrt{2\lambda}\phi_{\bar{K}}(x))\psi_K^L(\omega, x) \\ \psi_K^L(\omega, x) \end{pmatrix} e^{i\omega t} \quad (21)$$

for negative  $x$ . The function  $\psi_K^R(\omega, x)$  is the solution of the Schrödinger equation (6) with frequency  $\omega$  which vanishes for  $x \rightarrow \infty$ . Similarly,  $\psi_K^L(\omega, x)$  is the solution of (6) which vanishes for  $x \rightarrow -\infty$ . Further,  $\phi_K(x) = m/\sqrt{2\lambda} \tanh(x - L/2)$  for  $x > 0$  and  $\phi_{\bar{K}}(x) = -m/\sqrt{2\lambda} \tanh(x + L/2)$  for  $x < 0$ . For nonzero  $\omega$  continuity at the origin fixes  $\omega$

$$\omega^2 \psi_K^L \psi_K^R = - \left[ (\partial_x + c\sqrt{2\lambda}\phi_K) \psi_K^R \right] \left[ (\partial_x - c\sqrt{2\lambda}\phi_{\bar{K}}) \psi_K^L \right] \quad \text{at } x = 0 \quad (22)$$

The expressions for  $\psi_K^R$  and  $\psi_K^L$  are given by [1]

$$\begin{aligned} \psi_K^R(\omega, x) &= \exp -\kappa(x - \frac{L}{2}) \left[ -3 \tanh^2 \frac{m(x - \frac{L}{2})}{2} + 1 - \frac{4\kappa^2}{m^2} - \frac{6\kappa}{m} \tanh \frac{m(x - \frac{L}{2})}{2} \right] & \text{for } x > 0 \\ \psi_K^L(\omega, x) &= \exp \kappa(x + \frac{L}{2}) \left[ -3 \tanh^2 \frac{m(x + \frac{L}{2})}{2} + 1 - \frac{4\kappa^2}{m^2} + \frac{6\kappa}{m} \tanh \frac{m(x + \frac{L}{2})}{2} \right] & \text{for } x < 0 \end{aligned} \quad (23)$$

with  $\kappa = (m^2 - \omega^2)^{1/2} > 0$ . For large  $L$  the frequency  $\omega$  is small, and expanding (22) in powers of  $\omega^2$  one finds to leading order

$$\omega^2 = \frac{9}{4} m^2 \frac{(1 - \tanh \frac{mL}{4})^4}{(2 - \tanh \frac{mL}{4})^2} \sim 36m^2 \exp(-2mL) \quad (24)$$

Note that  $\omega^2$  tends to zero as  $\exp(-2mL)$  for large  $L$ . As  $\omega^2$  is positive, the frequencies in (20) and (21) are real.

The results for  $\omega^2 \approx 0$  for the bosons and fermions in a sphaleron background display a suggestive relation to the same results for a  $\bar{K}K$  background. For the bosons one finds  $\omega^2 = -48m^2e^{-mL}$  and  $\omega^2 = 0$  in a sphaleron background, but  $\pm 12m^2e^{-mL}$  in a  $\bar{K}K$  background. A tunneling argument shows that for the  $\bar{K}K$  system the two zero mode levels become split symmetrically around zero, yielding  $\omega^2 = \pm Ae^{-mL}$  (with  $A = 12m^2$ ). The factor  $e^{-mL}$  can be explained by considering the zero mode of the kink in a potential  $V + \Delta V$  where  $V$  is the potential of the kink and  $\Delta V$  is the potential due to the antikink. Perturbation theory yields then for the diagonal correction to  $H = \omega^2$  a value  $\int dx [\eta \Delta V \eta] \sim e^{-2mL}$ . However, the main effect is an off-diagonal mixing, yielding  $\omega^2 = \int dx [\bar{\eta} \Delta V \eta]$  where  $\bar{\eta}$  is the zero mode of the antikink. Diagonalization of the  $2 \times 2$  mixing matrix indeed produces  $\omega^2 \sim \pm e^{-mL}$ . In fact, we claim that the value of the splitting in  $\omega^2$  in the sphaleron case is  $-4A$ . The reason is that the magnitude of the splitting in the sphaleron background should be twice the magnitude of the splitting in the  $\bar{K}K$  background because in the sphaleron case forces work in both directions. Also the results for the fermionic distorted zero modes can be explained. The factor  $e^{-2mL}$  in  $\omega^2$  given by (24) for the  $\bar{K}K$  system is due to the overlap effect in  $\Delta H = \omega \sim \int \psi_1 \Delta V \psi_2 dx$  yielding  $\omega \sim e^{-mL}$ , (and thus  $\omega^2 \sim e^{-2mL}$ ), as  $\psi_1 \sim e^{-mL}$  where  $\Delta V$  is of order unity. This also explains why  $\omega^2$  for the fermions is positive.

The continuity of (20) and (21) at the origin  $x = 0$  requires  $\alpha = g\gamma$  and  $\beta = -g\delta$  where

$$g = \frac{i \left[ \partial_x - c\sqrt{2\lambda}\phi_{\bar{K}}(0) \right] \psi_K^L(\omega, 0)}{\omega \psi_K^R(\omega, 0)}. \quad (25)$$

The expression (20) can then be rewritten as

$$\begin{aligned} \psi(x, t) = g \left[ \gamma \left( \frac{\psi_K^R(\omega, x)}{\omega} (\partial_x + c\sqrt{2\lambda}\phi_K(x)) \psi_K^R(\omega, x) \right) e^{-i\omega t} \right. \\ \left. - \delta \left( -\frac{\psi_K^R(\omega, x)}{\omega} (\partial_x + c\sqrt{2\lambda}\phi_K(x)) \psi_K^R(\omega, x) \right) e^{i\omega t} \right] \quad \text{for } x \geq 0 \end{aligned} \quad (26)$$

while (21) is unchanged. Reality of  $\psi(x, t)$  implies  $\alpha = \beta^*$  and  $\gamma = \delta^*$ , while  $g^* = -g$ . This shows that we really have only one set of creation and annihilation operators,  $b = g\gamma$  and  $b^* = -g\delta$ . Notice also, that when the separation  $L$  is large, the expressions  $(\partial_x + c\sqrt{2\lambda}\phi_K(x))\psi_K^R(\omega, x)$  and  $(\partial_x - c\sqrt{2\lambda}\phi_{\bar{K}}(x))\psi_K^L(\omega, x)$  are very small (they vanish at  $\omega^2 = 0$  because of the BPS equation, and are of order  $\omega^2$  for nonzero  $\omega$ ). As a result, near  $x = L/2$  both solutions have mostly an upper component and a negligible lower component, while near  $x = -L/2$  the opposite holds. However near  $x = 0$  the upper and lower components of both solutions are of equal magnitude (although much smaller than the leading components at  $x = \pm L/2$ ).

From the equal-time canonical anticommutation relations of  $\psi(x, t)$  as given by the Dirac formalism for Majorana spinors  $\{\psi_i(x, t), \psi_j(y, t)\} = \delta_{ij}\delta(x - y)$  one reads off the anticommutators for the  $\gamma$  and  $\delta$ . One finds (after properly normalizing the wave functions for the almost-zero modes)

$$\{\gamma, \gamma\} = 0, \quad \{\delta, \delta\} = 0, \quad \{\gamma, \delta\} = 1 \quad (27)$$

Reality of  $\psi(x, t)$  implies

$$\gamma = \delta^\dagger \quad (28)$$

Shifting the origin in time from  $t = 0$  to  $t = \tau$  leads to operators  $\gamma(\tau) = e^{-i\tau\omega}\gamma$  and  $\delta(\tau) = e^{i\tau\omega}\delta$ . Independence of  $\tau$  implies  $\{\gamma(\tau), \delta(\tau)\} = 1$  and  $\{\gamma(\tau), \gamma(\tau)\} = \{\delta(\tau), \delta(\tau)\} = 0$ , which are obviously true.

For  $\omega \rightarrow 0$  ( $L \rightarrow \infty$ ) one can introduce two hermitian operators which commute with the Hamiltonian

$$\begin{aligned} b &= (e^{i\omega\tau}\gamma(\tau) + e^{-i\omega\tau}\delta(\tau)); & b &= b^\dagger, \quad b^2 = 1, \\ d &= \frac{1}{i}(e^{i\omega\tau}\gamma(\tau) - e^{-i\omega\tau}\delta(\tau)); & d &= d^\dagger, \quad d^2 = 1, \end{aligned} \quad (29)$$

The operators  $b$  and  $d$  are then the operators for the zero modes of the kink and antikink, respectively.

## 5 The isolated kink

Having understood mode counting for the kink-antikink system on the circle and on the infinite line, we now turn to the problem of the kink alone. This section consists of three parts: (i) it begins with an analysis of the three contributions to the energy density (localized near the kink, localized near the boundary or uniformly distributed), (ii) next we make explicit computations which corroborate the general analysis and (iii) we interpret the results in terms of a  $Z_2$  gauge symmetry.

### 5.1 Localized and delocalized energy

We shall begin with a very simple approach which is guaranteed to yield the correct answer for the kink mass [4]: given the solutions for the kink-antikink system with (for definiteness) periodic boundary conditions at  $x = \pm L$ , we just look at the behavior of these solutions halfway between kink and antikink, and this determines a set of boundary conditions at  $x = L$  and  $x = 0$  for a kink centered at  $x = \frac{L}{2}$ . If we use the corresponding frequencies in the mass formula, we must get the correct result, namely half the mass shift for the  $\bar{K}K$  system. These boundary conditions are periodic (P) or antiperiodic (AP) for the boson field fluctuations in both sectors, and also P and AP for fermions in the trivial sector. For fermions in the the kink sector, the conditions are twisted periodic (TP) or twisted antiperiodic (TAP) [specifically,  $\psi_1(-L/2) = \psi_2(L/2)$  and  $\psi_2(-L/2) = \psi_1(L/2)$  for TP,  $\psi_1(-L/2) = -\psi_2(L/2)$  and  $\psi_2(-L/2) = -\psi_1(L/2)$  for TAP].<sup>5</sup> Evidently, taking both P and AP conditions (or TP and TAP conditions) would overcount the number of states in the interval  $0 \leq x \leq L$  by a factor two, so one may take the contributions from each and then average the results.

In particular, by identifying the kink mass as half the  $\bar{K}K$  mass, one should take for the fermions the difference between the averages of mode sums with P and AP boundary

---

<sup>5</sup> Using  $\psi_1 = e^{ik(x+L/2)}$  on the far left,  $\psi_1 = e^{ik(x+L/2+\delta+\theta)}$  near  $x = 0$  and  $\psi_1 = Ae^{i(k(x-L/2)+2\delta+\theta)}$  on the far right (see figure 2) one finds from continuity at the origin  $A = e^{ikL}$ . The Dirac equation (7) yields  $\psi_2$ , in particular  $\psi_2(L) = e^{i(3kL/2+2\delta+3\theta/2)}$ . Imposing  $\psi_1(-L) = \psi_1(L)$  (which implies that also the derivative is periodic), one finds the quantization condition  $e^{i(2kL+2\delta+\theta)} = 1$ . Clearly  $\psi_2(L)/\psi_1(0) = e^{i(kL+\delta+\theta/2)} = \pm 1$ .

conditions in the trivial sector, and sums with TP and TAP boundary conditions in the kink sector. There is no fermionic P or AP solution with  $\omega^2 \sim 0$  in the trivial sector, and clearly therefore no corresponding fermion degree of freedom. (The general solution of the Schrödinger equation  $\psi_1 = \alpha e^{\kappa x} + \beta e^{-\kappa x}$  with  $\kappa^2 = m^2 - \omega^2 \sim m^2$  can not be P or AP for both  $\psi_1$  and  $\psi_2$ ). As there is only one fermionic zero-frequency TP or TAP **solution** in the kink sector (see below), there are no zero-frequency fermion **degrees of freedom** in either sector. However, in the kink-antikink system, there was one fermionic degree of freedom near zero frequency, see (26), hence the highest-energy mode in the kink sector must be counted with half the weight of any other mode. This may be accomplished, for example, by omitting the highest energy state in the sum for twisted antiperiodic boundary conditions<sup>6</sup>. Clearly this prescription violates the principle of summing equal numbers of modes, which is the basis of mode regularization.

The contradiction is resolved by noting that the mode regularization principle is based on fixed boundary conditions, so that one may think of the individual modes as ‘particle wave functions’ which are deformed in their  $x$ -dependence and shifted in frequency as the background classical field changes from trivial to kink. However, in the approach described above, the boundary conditions also change between the two sectors (from P and AP to TP and TAP). As noted by Goldstone and Wilczek for complex (Dirac) fermion fields, the 180° chiral rotation in (30) of the Yukawa coupling leads to a flow of 1/2 unit of fermion charge out of the region [12] (see figure 4).

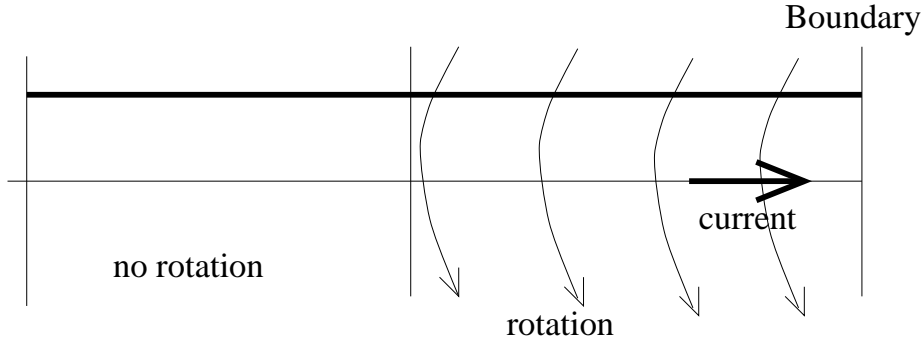


Figure 4. *The chiral rotation causes a current which leads to the accumulation of half a degree of freedom at the boundary if the boundary conditions are kept fixed.*

<sup>6</sup> For future use we do at this moment a little calculation. Having  $N$  modes for TP conditions in the kink sector and  $N - 1$  modes for TAP conditions in the kink sector, one finds that the difference of the mode sums is finite but nonzero. For  $N = 2M - 1$  one finds

$$\frac{1}{2} \sum \omega_{TP} - \frac{1}{2} \sum \omega_{TAP} = \frac{1}{2} \omega_{7,N} + \sum_{n=1}^{M-1} \omega_{7n} - \sum_{n=1}^{M-1} \omega_{8n} = \frac{\sqrt{\Lambda^2 + m^2}}{2} + \int_0^\Lambda \frac{dk}{2\pi} (-\pi) \omega' = \frac{m}{2}$$

(where we put  $\omega_7$  ( $\omega_8$ ) for the frequency with  $k$ , satisfying TP (TAP) conditions in the kink sector, which we define in our main text above equation (42)). Our arguments in the main text lead to the conclusion that one needs the average of the TP and TAP conditions, but it is clear from this little calculation that using only TP or TAP conditions yields an incorrect finite answer for  $M^{(1)}$ . Obviously, taking instead  $N - 1$  modes with TP conditions and  $N$  modes with TAP conditions leads to a divergent value for the difference (but the same result for the average).

In this way they gave a dynamical mechanism for the phenomenon which had been discovered by Jackiw and Rebbi [13], that a kink coupled to Dirac fermions carries half-integer fermionic charge. Fixed boundary conditions would stop this flow at the boundary, but it is obvious that simultaneously rotating the boundary conditions would maintain the flow. This certainly is consistent with the actual result in our case, that (on averaging between P and AP in the trivial sector, and TP and TAP in the kink sector) half a Majorana fermion mode must be omitted from the sum in the kink sector. Drawing a superficial analogy (which will become less superficial as we go on): having a complex Dirac fermion allows the same chiral rotation (and loss of half a unit of fermionic charge) as having two sets of boundary conditions for a real fermion (where the chiral rotation leads to the loss of half a degree of freedom).

The recipe obtained from the  $\bar{K}K$  system gives a reasonable interpretation for the kink alone, but does not satisfy the principle of mode regularization that one should use fixed boundary conditions. If we wish to use fixed boundary conditions, we run into the following problem: The periodic conditions in the trivial sector and twisted periodic conditions in the kink sector both are associated with ‘locally invisible’ boundaries (namely, with **plane wave** solutions). That is, they give no structure associated with the precise location of the boundary [3]. This means that, aside from the energy density localized around the kink, in principle the only other possible contribution would be a translationally invariant piece, corresponding to an energy density  $\mathcal{O}(1/L)$  as we shall explain.

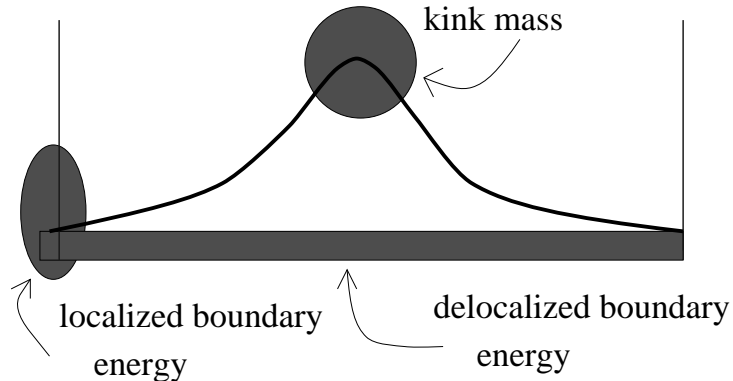


Figure 5. *The total energy density in the kink background.*

Except for this possibility, computing the quantum correction using these boundary conditions is identical in effect to the procedure advocated by Shifman et al. [6], to compute the local energy density and integrate it over the region of the kink. However, either periodic conditions in the kink sector or twisted periodic conditions in the trivial sector would produce a ‘visible’ boundary, forcing **standing wave** rather than plane wave solutions. In the fermionic sum  $\sum \omega_n^f$  there is then a true contribution to the mass and a boundary contribution  $E_{\text{boundary}}$ . Therefore, one obtains an energy shift  $M_{\text{kink}}^{(1)} - E_{\text{boundary}}$  if one uses fixed (i.e., in both sectors) P plus AP boundary conditions, and  $M_{\text{kink}}^{(1)} + E_{\text{boundary}}$  with fixed TP



(plus TAP) boundary conditions. By averaging the two forms one obtains  $\Delta M_{\text{kink}}$  by itself.

To spell this out further: P or AP conditions in the trivial sector have no localized boundary energy  $E_{\text{boundary}}$ . TP or TAP conditions in the kink sector also have no localized  $E_{\text{boundary}}$ . However, TP and TAP in the trivial sector, as well as P and AP in the kink sector, all have localized boundary energy  $E_{\text{boundary}}$ . Making a chiral rotation near the boundary which maps  $\phi \rightarrow -\phi$  and twists the fermions

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}' = e^{\pm i\frac{\pi}{2}\sigma_1} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \pm i \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}, \quad \phi' = e^{\pm i\pi} \phi \quad (30)$$

the localized boundary energy should not change<sup>7</sup>. (In addition, to keep the fermions real one needs a finite local gauge transformation  $\psi_1 \rightarrow i\psi_1$  and  $\psi_2 \rightarrow i\psi_2$ .) Thus  $P \rightarrow (TP, TAP)$  and  $AP \rightarrow (TAP, TP)$  and vice-versa.

Accepting that we need fixed boundary conditions (i.e. the same in the trivial and in the kink sector) for mode regularization, and observing that we need different boundary conditions to cancel the localized boundary energy, we are compelled to average over all four sets (P, AP, TP and TAP in both sectors) of boundary conditions for the fermions. The need for all four sets of boundary conditions to implement the chiral symmetry is similar to the need to consider all four spin structures for the string on a torus. The conclusion is that boundary energy occurs in the various cases as indicated in table I.

Sector	Boundary conditions	
	P and AP	TP and TAP
Trivial	0	$E_{\text{boundary}}$
Kink	$E_{\text{boundary}}$	0

Table I. *Localized boundary energy for different sectors and boundary conditions.*

Obviously, the boundary energy cancels in the average over all four choices with fixed boundary conditions.

What about the delocalized boundary energy? In the trivial sector with P and AP conditions, the difference of the P sum of  $\sqrt{\left(\frac{2\pi n}{L}\right)^2 + m^2}$  and the AP sum of  $\sqrt{\left(\frac{(2n+1)\pi}{L}\right)^2 + m^2}$  can be grouped into a sum over quartets of states, starting from the bottom. In each quartet the leading nonvanishing term is of order  $1/L^2$ , but summing over all modes, the total energy difference is of order  $1/L^8$ . Hence, one can forget in the computation of the kink mass the delocalized boundary energy in the trivial sector. This permits us to choose the average of the P sum and the AP sum as the energy of the trivial vacuum, which we define to be zero. In the kink sector one has locally invisible boundaries for TP and TAP conditions, so in these cases there could also be delocalized boundary energy. From the explicit calculation

<sup>7</sup> As the fermions are Majorana, one really should first complexify them (going to an  $N = 2$  model), but one can achieve the same goal by summing over both chiral rotations.

<sup>8</sup> Using  $\sqrt{\left(\frac{\pi(n+1)}{L}\right)^2 + m^2} - 2\sqrt{\left(\frac{\pi n}{L}\right)^2 + m^2} + \sqrt{\left(\frac{\pi(n-1)}{L}\right)^2 + m^2} = \left(\frac{\pi}{L}\right)^2 \frac{d^2\omega}{dk^2} + \dots$  with  $n = 2k + 1$ , one finds that the total energy difference is equal to  $\frac{\pi}{2L}$ .

that TP (with  $N$  modes) minus TAP (with  $N - 1$  modes) gives  $m/2$ , we conclude that there are different amounts of delocalized boundary energy in the TP and TAP sectors, so delocalized energy does, in fact, occur<sup>9</sup>. Nevertheless, returning to our  $\bar{K}K$  system, we notice that no delocalized energy could be created when one locally pulls the trivial configuration  $\phi = \mu/\sqrt{\lambda}$  down to the nontrivial configuration with a  $\bar{K}K$ , and then separates  $K$  and  $\bar{K}$ . Consequently, when the average is taken, the delocalized boundary energy contributions must cancel<sup>10</sup>. We summarize the results on delocalized boundary energy in table II.

Sector	Boundary conditions			
	P	AP	TP	TAP
Trivial	0	0	0	0
Kink	0	0	$\frac{\hbar m}{4}$	$-\frac{\hbar m}{4}$

Table II. *Delocalized boundary energy for different sectors and boundary conditions.*

What about the zero modes, and the correct counting of states in the sums? For fixed TP conditions, one has a single zero-frequency solution attached to the boundary in the trivial sector, namely (35) with  $a_1 e^{-mL/2} = a_2 e^{mL/2}$ , and a single one attached to the kink in the kink sector, namely (10) with  $a_1 / \cosh^2 \frac{mL}{2} = a_2 \cosh^2 \frac{-mL}{2}$ . Thus, no matter how one might weigh the contribution of such a single solution, the effect cancels exactly in the subtraction. For fixed P conditions, there are no zero modes in the trivial sector, and two in the kink sector, namely one attached to the kink and one attached to the boundary, given by (10) with  $a_1$  and  $a_2$  arbitrary. For TAP and AP conditions the same results hold<sup>11</sup>. Thus one must omit one (above-threshold) fermion mode from the Casimir sums over P and AP boundary conditions. Previously, from our study of the kink-antikink system, we were led to consider only TP and TAP conditions and then we needed to omit one term from their sum. Now we have a different message: we consider all four sets of P, AP, TP and TAP, and omit one term from the P sum and one term from the AP sum.

<sup>9</sup> We assume here that after subtracting the localized and the delocalized boundary energy, one obtains the true mass of the kink which should not depend on the boundary conditions [6]. Since in the TP and TAP sectors of the kink there is no localised boundary energy, the difference  $m/2$  must be due to delocalized boundary energy.

<sup>10</sup> These statements follow from a standard assumption in field theory, that in the presence of a mass gap, all correlators of observables fall exponentially with separation of arguments. Consider any correlator involving some number of factors  $\left(\phi^2(x) - \frac{\mu^2}{\lambda}\right)$  and one factor  $\epsilon(x)$ , the local energy density. By the general principle  $\langle \epsilon(x) \rangle$  must fall exponentially for  $x$  far from kink and from antikink. Thus, there can be no translationally invariant piece of the energy density. From our analysis of the kink-antikink system we know that the average over TP and TAP in the kink sector and P and AP in the trivial sector will not produce such a contribution. The novelty here is that for TP or TAP separately there is a finite difference, which must be attributed to a translationally invariant energy density. This fact suggests that some principle must require averaging over both TP and TAP corrections, excluding a delocalized energy of order  $m$ . We shall see shortly that there is indeed such a principle

<sup>11</sup> Actually, to obtain two solutions with AP conditions in the kink sector one needs exponentially small but nonvanishing  $\omega$ . One may start with the Schrödinger equation for  $\psi_1$  and raise  $\omega^2$  such that  $\psi_1$  vanishes at  $x = \pm L/2$ . Then the Dirac equation yields two solutions for  $\psi_2$  corresponding to  $\pm\omega$ , which are antisymmetric.

Averaging over the four cases of fixed nontwisted and twisted boundary conditions once again leads to a reduction in the kink sum for the fermions by half a fermion mode, but now the accounting is completely straightforward, unambiguous and canonically justified.

We have reached the following conclusions:

- (i) for fixed boundary conditions no change in the total number of degrees of freedom occurs, as expected,
- (ii) the issue how to weigh a single zero-frequency solution in the sums over modes need not be solved (although we shall solve it) because both the trivial and the kink sector have one such solution for twisted boundary conditions,
- (iii) In going from P and AP in the trivial sector to TP and TAP in the kink sector,  $1/2$  degree of freedom is lost on the average<sup>12</sup> (it is radiated away at the boundary, in agreement with [12]. When the boundary conditions change there is no reason to expect that the number of degrees of freedom remains the same).
- (iv) One must average over the four sets of boundary conditions for the following reasons (a) in order to get the correct answer (the same as from the  $\bar{K}K$  system), (b) the contributions from TP or TAP in the kink sector are different, so one may expect to need a particular combination of both, (c) the chiral rotation from one set of boundary conditions in the trivial sector links to two sets of boundary conditions in the kink sector, (d) in order that the localized boundary energy of table I cancels, and (e) in order that the delocalized boundary energy given in table II cancels.
- (v) For fixed boundary conditions, no degrees of freedom are lost, but now one zero-frequency solution can be attached to the boundary or to the kink. More specifically, for visible boundary conditions there is always one zero-frequency solution attached to the boundary, and when a kink is present there is always one zero-frequency solution attached to the kink. This yields four possible contributions, see table III.

Sector	Boundary conditions			
	P	AP	TP	TAP
Trivial	0	0	1	1
Kink	2	2	1	1

Table III. *Number of  $\omega \approx 0$  solutions for different sectors and boundary conditions.*

## 5.2 Explicit computations

We now give some details. We start with periodic boundary conditions for the bosonic fluctuations in the  $\bar{K}K$  system:  $\eta(-L) = \eta(L)$  and  $\eta'(-L) = \eta'(L)$ . We could use the sphaleron solution discussed before as background, or we could use  $\phi_{\bar{K}K}(x)$  to avoid elliptic functions at the price of not having a solution of the classical field equations at  $x = 0$ . For the fluctuations this makes negligible (exponentially small) difference.

---

<sup>12</sup> To avoid confusion, note that for the calculation of the kink mass one must take equal numbers of modes in the trivial and kink sector for each set of boundary conditions, but what these numbers are is not important, they may differ from one set to another.

The quantization condition for the double system is  $2kL + 2\delta(k) = 2\pi n$ ,  $-\infty < n < \infty$ . For the single kink we find two sets of boundary conditions: P and A

$$\begin{aligned} \eta(0) &= \eta(L) \quad \text{and} \quad \eta'(0) = \eta'(L) & \text{(P)} \\ \eta(0) &= -\eta(L) \quad \text{and} \quad \eta'(0) = -\eta'(L) & \text{(A)} \end{aligned} \quad (31)$$

With the solution  $\eta(x, t) \sim \exp(i[kx \pm \frac{1}{2}\delta(k)])$  we then find the quantization conditions

$$\begin{aligned} kL + \delta(k) &= 2n\pi & \text{(P)} \\ kL + \delta(k) &= (2n + 1)\pi & \text{(A)} \end{aligned} \quad (32)$$

where in both cases  $-\infty < n < \infty$ . Clearly we find for a single kink the same set of momenta in (32) as for the  $\bar{K}K$  system. The mass formula for a single kink then reads

$$\begin{aligned} M_b^{(1)} &= \frac{1}{4}\hbar \left[ 0 + \omega_B + m + 2 \sum_{n=2}^N \omega_n^P - \sum_{n=-N}^N \omega_n^{P,(0)} \right] \\ &+ \frac{1}{4}\hbar \left[ 0 + \omega_B + \sum_{n=-N}^{-2} \omega_n^A + \sum_{n=1}^{N-1} \omega_n^A - 2 \sum_{n=0}^{N-1} \omega_n^{A,(0)} \right] + \Delta M_b \\ &= \frac{\hbar\omega_B}{2} - \hbar m - \hbar \int_0^\Lambda \frac{dk}{2\pi} \omega' \delta + \Delta M_b \end{aligned} \quad (33)$$

We used that in the nontrivial periodic case the solution with  $n = 0$  is excluded, and the solutions with  $n = +1$  and  $n = -1$  both yield  $k = 0$ , giving the term  $m$ . In the nontrivial antiperiodic case the solutions with  $n = -1$  and  $n = 0$  are excluded. This yields (by construction) the correct mass for the bosonic kink. (In fact the P and AP conditions each give the same correct value for the bosonic fluctuations). In the  $\bar{K}K$  system we had two zero modes, and this corresponds to having one zero mode in each of the kink sectors.

Let us now turn to the fermionic fluctuations. In this case the issue of what set of boundary conditions to use is much more subtle. As explained earlier, the correct set of boundary conditions consists of periodic and antiperiodic, both twisted and untwisted, considered in both trivial and kink sectors. The formula for the mass correction from the fermions then reads

$$\begin{aligned} &+ \frac{1}{4}(P + A)_{\text{Trivial sector}} \\ &- \frac{1}{4}(P + A)_{\text{Kink sector}} \\ &+ \frac{1}{4}(TP + TAP)_{\text{Trivial sector}} \\ &- \frac{1}{4}(TP + TAP)_{\text{Kink sector}} \end{aligned} \quad (34)$$

As mentioned earlier, the first and the last lines appear naturally when one reduces the kink-antikink system to a single kink, so they do separately give the right result. Now we check that the full set of sums also gives the correct answer.

First, we address the issue of fermionic  $\omega = 0$  solutions under these boundary conditions. The formal solutions of the Dirac equation with  $\omega = 0$ , (8) and (9), are given by (10) for the case of the kink background and by

$$\psi_I = \begin{pmatrix} a_1 e^{-mx} \\ 0 \end{pmatrix}, \quad \psi_{II} = \begin{pmatrix} 0 \\ a_2 e^{mx} \end{pmatrix} \quad (35)$$

for the case of the trivial background  $\phi_{triv} = \frac{m}{\sqrt{2}\lambda}$ . Adjusting the coefficients  $a_1$  and  $a_2$ , one may try to satisfy the boundary conditions in (34). This results in the numbers of  $\omega \approx 0$  solutions in particular sectors given by table III.

Next we consider the continuous spectrum. The quantization conditions for P and AP boundary conditions in the trivial sector are obvious.

We now address the TP conditions in the trivial sector. If one puts

$$\psi_1 = e^{ikx} + ae^{-ikx} \quad (36)$$

then it follows from the Dirac equation (7) that

$$\psi_2 = e^{i(kx + \frac{\theta}{2})} - ae^{-i(kx + \frac{\theta}{2})} \quad (37)$$

where we define  $\theta$  such that  $e^{i\frac{\theta}{2}} = -\frac{k}{\omega} + i\frac{m}{\omega}$ . Twisted periodic conditions read  $\psi_1(0) = \psi_2(L)$  and  $\psi_2(0) = \psi_1(L)$ . We plug in  $\psi_{1,2}$  and solve for  $a$ ; this gives the following quantization condition

$$\frac{e^{i(kL + \frac{\theta}{2})} - 1}{e^{-i(kL + \frac{\theta}{2})} + 1} = a = \frac{e^{i\frac{\theta}{2}} - e^{ikL}}{e^{-ikL} + e^{-i\frac{\theta}{2}}} \quad (38)$$

which can be rewritten as  $\sin kL = 0$ , i.e.  $kL = \pi n$ . Notice that if one changes  $k \rightarrow -k$  in (36) and (37), then  $e^{i\frac{\theta}{2}} \rightarrow -e^{-i\frac{\theta}{2}}$ , so that  $a \rightarrow \frac{1}{a}$ , and (36) and (37) stay the same up to normalization. Therefore negative  $k$  do not produce new independent solutions. There is formally a solution with  $n = 0$ , i.e.  $k = 0$ , but for this solution (38) yields  $a = -1$  and then (36) and (37) yield  $\psi_1 = \psi_2 = 0$  everywhere, so we must also exclude  $n = 0$ .

For TAP conditions in the trivial sector one gets the same result (which is obvious: one changes the sign of  $\psi_2$  and then requires twisted periodic conditions on  $\psi_1$  and  $-\psi_2$ ).

The P boundary conditions in the kink sector correspond to standing wave solutions of the Dirac equation (7) which were found in [2]. The general solution of the Dirac equation for  $\psi_1$  reads

$$\begin{aligned} \psi_1(x) = & \exp ik(x - \frac{L}{2}) \left[ -3 \tanh^2 \frac{m(x - \frac{L}{2})}{2} + 1 + \frac{4k^2}{m^2} + \frac{6ik}{m} \tanh \frac{m(x - \frac{L}{2})}{2} \right] \\ & + a \exp -ik(x - \frac{L}{2}) \left[ -3 \tanh^2 \frac{m(x - \frac{L}{2})}{2} + 1 + \frac{4k^2}{m^2} - \frac{6ik}{m} \tanh \frac{m(x - \frac{L}{2})}{2} \right] \end{aligned} \quad (39)$$

which gives the following asymptotic expressions for fermion components

$$\psi_1 = \begin{cases} -e^{i(kx - \frac{\delta}{2})} - ae^{-i(kx - \frac{\delta}{2})}, & x \approx 0 \\ -e^{i(kx + \frac{\delta}{2})} - ae^{-i(kx + \frac{\delta}{2})}, & x \approx L \end{cases} \quad (40)$$

and, from the Dirac equation (7),

$$\psi_2 = \begin{cases} e^{i(kx - \frac{\delta}{2} - \frac{\theta}{2})} - ae^{-i(kx - \frac{\delta}{2} - \frac{\theta}{2})}, & x \approx 0 \\ e^{i(kx + \frac{\delta}{2} + \frac{\theta}{2})} - ae^{-i(kx + \frac{\delta}{2} + \frac{\theta}{2})}, & x \approx L \end{cases} \quad (41)$$

Requiring periodicity, it is clear<sup>13</sup> that it can be achieved either for  $a = 1$ ,  $kL + \delta + \theta = 2\pi n$  or for  $a = -1$ ,  $kL + \delta = 2\pi n$ , and only positive  $n$  are needed to produce distinct solutions. In particular for  $n = 1$ ,  $k = 0$  one computes from the latter set  $\psi_1|_{k=0} = \left[1 - 3 \tanh^2 \frac{m(x-L/2)}{2}\right] (1 + a) = 0$ , so by the Dirac equation  $\psi_2 = 0$ . Thus this solution must be excluded, just like the  $k = 0$  solution in TP of the kink sector.

The situation with AP conditions in the kink sector is quite analogous. The antiperiodicity is achieved by putting  $a = -1$ ,  $kL + \delta + \theta = 2\pi n + \pi$  or by  $a = 1$ ,  $kL + \delta = 2\pi n + \pi$ . In particular there seems to be a solution at  $n = 0$ ,  $k = 0$  in the first set. This solution is excluded by the same argument as before: it is easy to check that in this case  $\psi_1 = \psi_2 = 0$  everywhere.

The TP conditions in the kink sector were worked out in [3]; the computation for the TAP case in the kink sector is again straightforward (it turns out that these two sets of boundary conditions are the only ones consistent with plane wave solutions, see [3]).

Thus, the quantization conditions for fermions are:

## I. Nontwisted

- 1) Periodic trivial sector:  $kL = 2\pi n$ , all  $n$ .
- 2) Antiperiodic trivial sector:  $kL = 2\pi n + \pi$ , all  $n$ .
- 3) Periodic kink sector: a)  $kL + \delta + \theta = 2\pi n$ ,  $n = 1, 2, 3, \dots$  and b)  $kL + \delta = 2\pi n$ ,  $n = 2, \dots$ .
- 4) Antiperiodic kink sector: a)  $kL + \delta + \theta = 2\pi n + \pi$ ,  $n = 1, 2, \dots$  and b)  $kL + \delta = 2\pi n + \pi$ ,  $n = 1, 2, \dots$ .

## II. Twisted

- 5) Twisted periodic trivial sector: a)  $kL = 2\pi n$ ,  $n = 1, 2, \dots$  and b)  $kL = 2\pi n + \pi$ ,  $n = 0, 1, 2, \dots$ .
- 6) Twisted antiperiodic trivial sector: a)  $kL = 2\pi n$ ,  $n = 1, 2, \dots$  and b)  $kL = 2\pi n + \pi$ ,  $n = 0, 1, 2, \dots$ .
- 7) Twisted periodic kink sector:  $kL + \delta + \frac{\theta}{2} = 2\pi n$ , all  $n$ ,  $n \neq 0$ .
- 8) Twisted antiperiodic kink sector:  $kL + \delta + \frac{\theta}{2} = 2\pi n + \pi$ , all  $n$ ,  $n \neq 0, -1$ .

We now work out the mass corrections due to fermions in the kink background for each fixed set of boundary conditions separately. In all the sums we keep equal numbers of modes in both trivial and nontrivial sectors. (For these fermionic corrections we subtract the contributions in the kink sector from the contributions in the trivial sector.)

We start with P boundary conditions. The fermions give the contribution

$$\begin{aligned}
M_f^{(1)}(P) &= \frac{\hbar}{2} \sum_{n=-N}^N \omega_{1)} - \frac{\hbar}{2} \sum_{n=1}^N \omega_{3a)} - \frac{\hbar}{2} \sum_{n=2}^N \omega_{3b)} - 0 - \frac{\hbar\omega_B}{2} + \Delta M_f \\
&= -\frac{\hbar\omega_B}{2} + \hbar m + \hbar \int_0^\Lambda \frac{dk}{2\pi} \omega' \left( \delta + \frac{\theta}{2} \right) + \Delta M_f
\end{aligned} \tag{42}$$

where we have taken into account one  $\omega = 0$  fermionic degree of freedom in the kink sector,

---

<sup>13</sup> Alternatively, an explicit calculation similar to (38) yields  $\sin\left(\frac{kL+\delta}{2}\right) = 0$  or  $\sin\left(\frac{kL+\delta+\theta}{2}\right) = 0$ .

see table III. For AP boundary conditions one obtains

$$\begin{aligned}
M_f^{(1)}(AP) &= \hbar \sum_{n=0}^N \omega_{2n} - \frac{\hbar}{2} \sum_{n=1}^N \omega_{4a} - \frac{\hbar}{2} \sum_{n=1}^N \omega_{4b} - 0 - \frac{\hbar \omega_B}{2} + \Delta M_f \\
&= -\frac{\hbar \omega_B}{2} + \hbar m + \hbar \int_0^\Lambda \frac{dk}{2\pi} \omega' \left( \delta + \frac{\theta}{2} \right) + \Delta M_f
\end{aligned} \tag{43}$$

and again, one  $\omega \approx 0$  fermionic degree of freedom in the kink sector is counted in the sum. The result is exactly the same as in the case of P boundary conditions.

For TP one does not have an  $\omega \approx 0$  degree of freedom, so

$$\begin{aligned}
M_f^{(1)}(TP) &= \frac{\hbar}{2} \sum_{n=1}^N \omega_{5a} + \frac{\hbar}{2} \sum_{n=0}^N \omega_{5b} - \hbar \sum_{n=1}^N \omega_{7n} - \frac{\hbar \omega_B}{2} + \Delta M_f \\
&= \frac{\hbar \sqrt{\Lambda^2 + m^2}}{4} - \frac{\hbar \omega_B}{2} + \frac{\hbar m}{4} + \hbar \int_0^\Lambda \frac{dk}{2\pi} \omega' \left( \delta + \frac{\theta}{2} \right) + \Delta M_f
\end{aligned} \tag{44}$$

For TAP, however, we get a result which is different from the result (44) for TP:

$$\begin{aligned}
M_f^{(1)}(TAP) &= \frac{\hbar}{2} \sum_{n=1}^N \omega_{6a} + \frac{\hbar}{2} \sum_{n=0}^N \omega_{6b} - \hbar \sum_{n=1}^N \omega_{8n} - \frac{\hbar \omega_B}{2} + \Delta M_f \\
&= -\frac{\hbar \sqrt{\Lambda^2 + m^2}}{4} - \frac{\hbar \omega_B}{2} + \frac{3\hbar m}{4} + \hbar \int_0^\Lambda \frac{dk}{2\pi} \omega' \left( \delta + \frac{\theta}{2} \right) + \Delta M_f
\end{aligned} \tag{45}$$

Actually, only the averages  $(P + AP)/2$  and  $(TP + TAP)/2$  are invariant under the  $Z_2$ -gauge symmetry  $\psi \rightarrow -\psi$ . It is easy to compute these averages, which of course do not have linear divergences. In terms of  $M_f^{(1)}$ , the fermionic contribution to the mass of the kink in (1), they can be written as

$$\frac{1}{2} [M_f^{(1)}(P) + M_f^{(1)}(AP)] = M_f^{(1)} + \frac{\hbar m}{4} \tag{46}$$

$$\frac{1}{2} [M_f^{(1)}(TP) + M_f^{(1)}(TAP)] = M_f^{(1)} - \frac{\hbar m}{4} \tag{47}$$

Note the presence of a ‘half-mode’ contribution  $\frac{\hbar m}{4}$  representing the boundary energy, which as expected appears with opposite signs in the  $(P+AP)$  and  $(TP+TAP)$  sums. To find out whether this boundary energy is localized or not, we consider the difference of the mode densities of  $TP+TAP$  minus  $P+AP$  in the trivial sector<sup>14</sup>. One computes for the density

---

<sup>14</sup> In the  $P+AP$  sector there is no localized boundary energy, and the delocalized boundary energy density is proportional to  $1/L^2$ . Hence, from  $P+AP$  we get no boundary energy at all, but the reason we subtract it from the  $TP+TAP$  is to make the result convergent.  $P+AP$  really defines the energy of the trivial vacuum.

difference of continuum modes with TP+TAP versus P+AP boundary conditions<sup>15</sup>

$$\frac{1}{2} [\{\rho_{TP}(x) + \rho_{TAP}(x)\}_{cont} - \{\rho_P(x) + \rho_{AP}(x)\}_{cont}] = - \int_0^\infty \frac{dk}{\pi} \frac{m^2 \cos(2kx)}{k^2 + m^2} = -\frac{m}{2} e^{-2m|x|}, \quad (48)$$

so that the mode density is localized around the boundary, and leads to a net mode number shift  $\delta n = -\frac{1}{2}$ . This computation for Majorana fermions in the trivial sector is equivalent to the result of Jackiw and Rebbi [13] for Dirac fermions in the presence of a kink. Indeed, a little thought shows that fermions in the trivial sector ‘feel’ the twisted periodic boundary conditions as equivalent to a kink (or antikink) of zero width.

Instead of the mode density, we may also compute the energy density. Using  $\int_0^\infty \frac{\cos(ak)dk}{\sqrt{k^2+m^2}} =$

---

<sup>15</sup> First, we find the proper normalization for the continuum modes. From (38) one finds for TP and TAP conditions in the trivial sector  $kL = \pi n$  and then (38) yields  $a$ , and (36) and (37) yield  $\psi_1$  and  $\psi_2$

$$\begin{aligned} \psi_1 &= A \left\{ \left[ e^{-i\frac{\theta}{2}} + (-1)^n \right] e^{ikx} + \left[ e^{i\frac{\theta}{2}} - (-1)^n \right] e^{-ikx} \right\} \\ \psi_2 &= A \left\{ \left[ e^{-i\frac{\theta}{2}} + (-1)^n \right] e^{i(kx+\frac{\theta}{2})} - \left[ e^{i\frac{\theta}{2}} - (-1)^n \right] e^{-i(kx+\frac{\theta}{2})} \right\} \end{aligned}$$

with  $k = \frac{\pi n}{L}$  and  $\cos \frac{\theta}{2} = -\frac{k}{\omega}$ ,  $\sin \frac{\theta}{2} = \frac{m}{\omega}$ . Here  $A$  is a constant which we will fix for the normalization. For the absolute values one obtains

$$\begin{aligned} |\psi_1|^2 &= |A|^2 \{4 + 2 \cos[2kx - \theta] - 2 \cos[2kx]\} \\ |\psi_2|^2 &= |A|^2 \{4 + 2 \cos[2kx + \theta] - 2 \cos[2kx]\} \end{aligned}$$

and the density of the  $n^{\text{th}}$  mode, normalized to unity, is

$$|\psi_1|^2 + |\psi_2|^2 = \frac{1}{L} \left\{ 1 + \frac{1}{2} \cos[2kx] (\cos \theta - 1) \right\}$$

Using also the expression for the zero mode (35), one gets for the total density in the trivial sector with TP and TAP conditions

$$\begin{aligned} \rho_{TP} + \rho_{TAP} &= 2 \times (|\psi_I|^2 + |\psi_{II}|^2) + 2 \times \sum_n [|\psi_{n,1}|^2 + |\psi_{n,2}|^2] \\ &= 2 \times m \frac{e^{-2mx} + e^{-2m(L-x)}}{1 - e^{-2mL}} + \frac{1}{L} \sum_{n=1}^N [2 + \cos(2kx)(\cos \theta - 1)] \end{aligned}$$

where the first term stands for the two  $\omega = 0$  solutions. For the P and AP conditions in the trivial sector we get simply  $\rho_P + \rho_{AP} = \frac{1}{L} \left( 1 + \sum_{n=1}^N 2 \right)$ . Neglecting  $e^{-mL}$  terms and identifying  $x = 0$  with  $x = L$  as the boundary at  $x = 0$  for  $-L/2 \leq x \leq L/2$ , we get for the difference of TP+TAP and P+AP

$$\begin{aligned} \{\rho_{TP} + \rho_{TAP}\} - \{\rho_P + \rho_{AP}\} &= 2me^{-2m|x|} - \frac{1}{L} + \frac{1}{L} \sum_{n=1}^N \frac{-2m^2 \cos(2\frac{\pi n}{L}x)}{(\frac{\pi n}{L})^2 + m^2} \\ &= 2me^{-2m|x|} - \frac{1}{L} + \frac{1}{2} \left[ \frac{2}{L} - \int_{-\infty}^{\infty} \frac{dk}{\pi} \frac{2m^2 \cos(2kx)}{k^2 + m^2} \right] \\ &= 2me^{-2m|x|} - me^{-2m|x|} = me^{-2mx}. \end{aligned}$$



$K_0(am)$  (where  $K_0(x)$  is the modified Bessel function) to obtain for the energy density

$$\epsilon_{TP+TAP}(x) - \epsilon_{P+AP}(x) = -\hbar \frac{m^2}{\pi} K_0(2m|x|) \quad , \quad (49)$$

which is also localized around the boundary, and leads to a net boundary energy for twisted periodic boundary conditions in the trivial sector,  $\delta M_{bound} = \hbar m/4$ . This proves that there is no delocalized energy in (46) and (47), in complete accord with the principle of cluster decomposition.

The average of (42-45) or (46-47) gives

$$\frac{1}{4} [M_f^{(1)}(P) + M_f^{(1)}(AP) + M_f^{(1)}(TP) + M_f^{(1)}(TAP)] = M_f^{(1)} \quad (50)$$

Adding this result to (33), one recovers the correct result for the susy kink  $M_s^{(1)}$ , namely (1).

With these results one may address the discrepancy for  $M^{(1)}$  obtained by mode regularization between [2] and the accepted value (1). In [2] eq. (60) the authors computed the one-loop correction to the energy using mode regularization for fixed periodic boundary conditions. Thus they should have obtained  $M^{(1)} + \hbar m/4$  (where the latter term is the localized boundary energy for periodic conditions in the kink sector, as indicated in (46)), and they did. Their calculation was a correct application of mode regularization, but gave the total effect of changing from trivial to kink background with fixed periodic boundary conditions, including the boundary energy which is not part of the localized quantum correction to the mass of the kink.

This brings up another question: Why do the methods of [3] (where  $\frac{d}{dm} \sum \omega$  was first evaluated) and of [4] (energy cut-off using a smooth interpolating function) get the accepted answer for the mass of the kink, even though in these methods no information is used about the bottom continuum modes, as in mode regularization? Answer: The boundary condition in these works do not change the density of states. Therefore, a formula depending only on the density of states through the phase shift (as [3] and [4] do) will give an answer for the mass independent of the boundary conditions. Because that formula agrees with the result from the  $\bar{K}K$  system, which has no boundary energy, it should be correct regardless of boundary conditions. What is lost by these methods, and this might well be described as an advantage, is the calculation of the total energy, including the boundary energy. Clearly, this type of regulation gives correctly the **local** energy density associated with the boundary, if there is such, because the local density is insensitive to  $O(1/L)$  contributions from the bottom of the spectrum. By conservation of energy, therefore, what it does not necessarily give correctly is the delocalized energy associated with the boundary. Put differently, in this type of scheme there is no reliable information about the delocalized energy, but there is reliable information about the mass of the kink, embodied in a single, global integral. By invoking the principle of cluster decomposition, with its implication that in a correct calculation there cannot be any delocalized energy, one may circumvent even the one disadvantage of these schemes. However, as we have seen, with mode regulation one may check the principle directly.

### 5.3 The $Z_2$ gauge symmetry

It still remains to show that a single solution with  $\omega = 0$  for fermions in a given sector with given boundary conditions does not correspond to any degree of freedom at all, and also to discuss the effect of such a solution on the Hilbert space. In the mode expansion of the Majorana field, the coefficient  $c_0$  of the zero mode is a single, idempotent Hermitian operator. This follows from the equal-time canonical anticommutation relations. The ground state may be chosen as an eigenstate<sup>16</sup> of  $c_0$ , so  $|ground\rangle = \frac{1}{2}(1 + c_0)|\Omega\rangle$ . Consequently, all states in the Hilbert space may be obtained by the action of local operators on  $|ground\rangle$ . No such operator would connect  $\frac{1}{2}(1 + c_0)|\Omega\rangle$  with  $\frac{1}{2}(1 - c_0)|\Omega\rangle$ . For  $c_0$  this is true by construction, but more complicated operators either have one factor  $c_0$  or no factor  $c_0$ , and in both cases one never leaves the half of the Hilbert space one is in. The other half of the Hilbert space is a copy of the first under the action of the discrete  $Z_2$  symmetry which maps fermion fields  $\psi$  into  $-\psi$ . The  $Z_2$  symmetry is actually a discrete gauge symmetry because it leaves all observables ( expectation values of operators which contain an even number of fermion fields ) invariant, just as in quantum mechanics phase factors of a continuous  $U(1)$  symmetry multiplying state vectors are not observable.

The  $Z_2$  symmetry  $\psi \rightarrow -\psi$  is hidden: That is, the kink ground state  $(1 + c_0)|\Omega\rangle$  is not manifestly invariant under it ( $c_0$  is mapped into  $-c_0$  under  $Z_2$ ). A better way of defining the ground state might be to say that it consists of a set with the two elements  $(1 + c_0)|\Omega\rangle$  and  $(1 - c_0)|\Omega\rangle$ . The state  $(1 + c_0)|\Omega\rangle$  is then simply a representative. Clearly, with this definition the ground state is  $Z_2$  gauge invariant and unique. On the other hand, it has been observed by Ritz et al. [14] that the ground state is not annihilated by the (linearized) supersymmetry generator  $Q_2$ . Rather it is mapped into itself, because  $Q_2$  is proportional to  $c_0$ <sup>17</sup>. Thus, half of the supersymmetry is spontaneously broken, as has long been known, but at the same time the unbreakable  $Z_2$  gauge symmetry only is hidden, i.e., not manifest.

The necessity of averaging over more than one set of boundary conditions implies a refinement of the assertion by Shifman et al. [6] that boundary conditions are unimportant if one computes the energy by calculating a regulated, renormalized energy density and integrating this density only in the region of the kink. One might expect the error in this calculation to be exponentially small, associated with exponential localization of the boundary energy. As discussed earlier, with the appropriate averaging over boundary conditions the total delocalized energy vanishes, which is clear because for the  $\bar{K}K$  system there is no delocalized energy. Explicit calculation shows that the difference in the trivial sector between the AP and P contributions to the delocalized energy (the only kind there is in this case), is of order  $1/L$ . Thus one can indeed forget about delocalized energy from the trivial sector. However, the difference  $m/2$  between sums for TP and TAP conditions in the kink sector represents a translationally invariant contribution to the energy, which would imply a spurious finite shift in the energy of the kink.

Thus the assertion in [6] that the kink mass may be calculated by integrating only over,

---

<sup>16</sup> At first sight it may seem strange to have a ground state which is half fermionic and half bosonic, but in 2 dimensions there is less distinction between fermions and bosons.

<sup>17</sup> Using  $Q = \int [\partial_\mu \phi \gamma^\mu \gamma^0 \psi + U \gamma^0 \psi] dx$  one finds  $Q_2 \sim \int (\partial_x \phi_K) \psi_1 dx$ . Since  $\partial_x \phi_K$  is proportional to the zero mode in  $\psi_1$ , see (10), and orthogonal to the nonzero modes, one finds  $Q_2 \sim c_0$ .

say, the half-space surrounding the kink, and staying well away from the boundary, indeed is correct, but with the proviso that boundary conditions which provide infrared regulation in this calculation must respect the  $Z_2$  gauge symmetry: All of the boundary conditions considered above may be visualized as jump conditions for wavefunctions defined on a circle. The effect of introducing a  $Z_2$  flux through that circle when a kink is present would be to interchange TP and TAP conditions, but we have seen that there is one less mode just above the mass threshold for TAP than for TP. Because the coupling to this flux is a discrete form of a continuous gauge symmetry, a change in the number of states would in the continuous version for complex fermion fields correspond to the abrupt disappearance of a unit of conserved charge, constituting an anomalous violation of the gauge invariance. To prevent such an anomaly, it is necessary and sufficient to use a regulation which preserves the gauge invariance, namely, describing the system as an incoherent, equal, superposition of TP and TAP boundary conditions. It is for this reason that in the discussion of the energy sums involving four different boundary conditions we have further bundled the sums into pairs, (TP + TAP) (essential bundling) and (P + AP) (allowed but not essential bundling).

There is another  $Z_2$  symmetry of the action, the transformation  $\phi \rightarrow -\phi$ , and simultaneously the twist  $\psi \rightarrow \pm\sigma_1\psi$ . This transformation imposed as a jump condition is locally invisible, but globally changes a system whose lowest state is the trivial vacuum to one whose lowest state is a kink, or more precisely, a half-sphaleron: For a circle of circumference  $2L$  with a sphaleron in metastable equilibrium, join any two points, making a circle of circumference  $L$ . The matching or boundary conditions on the half-sphaleron are precisely those of the second  $Z_2$  symmetry, and because the lowest energy configuration in this domain must be a kink (or equally well, an antikink), slightly squeezed because it is on a circle, it is obvious that the field is precisely that of half the sphaleron. The main, if not the only difference, from the full sphaleron is that there is no instability, because there is no possibility of kink-antikink annihilation in the presence of the jump condition. As with the sphaleron, if  $L$  is too small, then the lowest solution is simply  $\phi_{\text{classical}} = 0$ . However, for larger  $L$ , the half-sphaleron becomes absolutely stable. It is, however, not a BPS solution even at the classical level, because the BPS bound can only be saturated on the infinite line<sup>18</sup>.

This discussion complements a recent analysis by Binosi et al. [15] of solitons with winding number  $\pm 1$  in an  $N = 2$  supersymmetric theory with a potential depending on a complex  $\phi$  and periodic in  $\Re\phi$ . It is clear that there is quantum tunneling between soliton and antisoliton in both cases. However, their  $N = 2$  soliton is quantum unstable but classically saturates the BPS bound, whereas our kink is quantum stable but already violates the BPS bound at the classical level. A tentative conclusion from these two examples is that tunneling between soliton and antisoliton is likely to be the most generic feature for such systems.

We now are in a position to address an issue glossed over above: How general is the statement that the methods of [2] and [3] are independent of boundary conditions? One conspicuous case for which these methods do not work, as mentioned already in the introduction, is that of ‘supersymmetric’ boundary conditions  $\phi(0) = \psi_1(0) = \phi(L) = \psi_1(L) = 0$ . There are two reasons for this failure. First, the mentioned Dirac spinor conditions are equivalent

---

<sup>18</sup> The BPS equation  $\partial_x\phi + U(\phi) = 0$  at the point where  $\phi$  is maximal (and thus  $\partial_x\phi = 0$ ) requires that  $U(\phi) = 0$ , but then the only solution is  $\phi = m/\sqrt{2\lambda}$ .

to energy-dependent boundary conditions in a Schrödinger equation, and hence do change the density of states. Secondly, this choice of boundary conditions obviously violates the second  $Z_2$  symmetry, under which  $\psi_1(0) = \psi_1(L) = 0$  in the kink sector  $\rightarrow \psi_1(0) = \psi_2(L) = 0$  in the trivial sector. To average over a set of fixed boundary conditions while respecting the  $Z_2$  symmetry, one must have both these choices in both sectors. Indeed, doing so will remove the logarithmic divergence mentioned earlier, and also will reproduce the change in density of states between trivial and kink sectors due to the kink alone. (The details for these boundary conditions will be worked out explicitly elsewhere [16]). Not surprisingly, the logarithmically divergent energy associated with violating the second  $Z_2$  symmetry corresponds to a delocalized energy density, which gives another reason why such a choice of boundary condition should not be allowed. That this energy is delocalized is evident from the fact that the change in density of states comes from the energy dependence of the boundary conditions, which simply spaces continuum levels differently ‘by fiat’, so that there is no position where the associated energy shift should be localized.

Should both these discrete symmetries be considered gauge symmetries? As we have seen already, the second  $Z_2$  is similar to the first in that each one when applied to a jump condition has global implications. The first introduces a half-quantum of flux through the circle on which the fields are defined, while the second converts the sector with kink number 0 mod 2 into the sector with kink number 1 mod 2. Nevertheless, while globally significant, both these jump conditions may be applied at any point, and there will be no local consequences at that point. However, there are two important reasons not to describe the second  $Z_2$  as also a gauge symmetry. 1) If one goes from real fields to complex fields, so that  $Z_2$  becomes  $U(1)$ , then the first corresponds to standard gauge coupling, but the second would correspond to axial gauge coupling. It is well known that this  $U(1) \times U(1)$  is an anomalous theory. 2) In the sector with the second  $Z_2$  jump condition, as mentioned earlier it is possible to describe paths in field configuration, with action proportional to the circumference of the circle, connecting kink and antikink. This would make no sense if kink and antikink were identical, as would be implied by treating the second  $Z_2$  as a gauge symmetry. One may see this also by considering the sphaleron configuration, in which there may be a convention used to define the kink, but the distinction between kink and antikink is clear from the fact that they can annihilate each other. Indeed, the same point is manifest already for the vacuum configuration  $\eta = \pm \text{constant}$ . For a finite-circumference ring, there will be tunneling between the positive and negative values, leading to two nearly degenerate ground states which are equal superpositions of the two values. Only for infinite circumference do we have the thermodynamic limit in which spontaneous symmetry breaking occurs, and the two values are completely independent.

## 6 Conclusions

Mode regularization, i.e., the simple prescription that one subtracts vacuum energies for the same number of modes with and without some background, differs from many other regularization schemes in that the cutoff parameter need not be averaged over some continuous weight function which goes from unity at low energies to zero at high energies, as is necessary

in particular for energy or momentum cutoff [4]. This attractive feature gives a strong incentive to investigate whether the scheme is universally applicable. Here we have studied mode regularization for the case of the kink in (1+1) dimensions, including Majorana fermions. There is a well-known subtlety in counting boson zero modes, that zero-frequency modes must be expressed as collective coordinates, so that for each coordinate there is a conjugate momentum, giving rise to raising and lowering operators just like those for nonzero frequencies. Thus the two bosonic zero modes of a widely separated kink-antikink system become two collective coordinates, and they correspond to two pairs of  $(P, X)$  variables, counting as two degrees of freedom. This makes sense because it keeps the total number of modes constant as the corresponding squared frequency  $\omega^2$  goes to or through zero.

For fermions the situation is less familiar. By studying the problem in three closely related systems, kink-antikink, sphaleron (i.e., kink and antikink symmetrically placed on a circle), and isolated kink, we find a very different behavior from that for bosons, namely, the number of  $\omega \sim 0$  degrees of freedom (one) in the  $\bar{K}K$  system is half the number (two) of fermionic  $\omega \sim 0$  solutions: The two zero modes in the Dirac equation for the fermions together give one annihilation and one creation operator in the expansion of the fermion field, hence one degree of freedom.

This suggests that for an isolated kink, there is one boson zero mode degree of freedom, but only half a fermion zero mode degree of freedom. This half is interpreted as being due to different boundary conditions for which the energies must be averaged to give the correct mass shift. Two of the boundary conditions (periodic twisted and antiperiodic twisted) give a single zero mode in the Dirac equation, and the hermitian coefficient  $c_0$  of this mode function in the fermion field leaves the ground state invariant. There is no doubling of the Hilbert space due to this  $c_0$  because the states  $(1 + c_0)|\Omega\rangle$  and  $(1 - c_0)|\Omega\rangle$  are equivalent under the  $Z_2$  symmetry  $\psi \rightarrow -\psi$ .

The conclusion that the ground state is an eigenstate of a fermionic operator is at first thought puzzling. It not only violates intuition based on widespread experience, but also appears to contradict the well-known superselection rule forbidding coherent superposition of states with even and odd fermion number. Although Majorana fermions do not carry an additive, conserved fermion number  $F$ , still the fermion field anticommutes with the  $Z_2$  factor  $(-1)^F$ . However, in one space dimension the distinction between bosons and fermions is not pronounced, because there is no spin connected to the statistics; for example, one can bosonise fermions in string theory. The susy multiplet also is unusual: it contains 2 states for the non-BPS case [4], but only one state for the BPS case, as discussed in [6] and this article. Thus, the state  $(1 + c_0)|\Omega\rangle$  is not an exception to the superselection rule, but rather a unique and unexpected illustration of that rule.

The  $Z_2$  symmetry  $\psi \rightarrow -\psi$  is actually a gauge symmetry because it leaves all possible observables invariant. That is, there is no conceivable field which could be added to the Lagrangian as a perturbation and would give a coupling not invariant under  $\psi \rightarrow -\psi$ . Thus there is no zero-frequency fermionic degree of freedom for these boundary conditions, and consequently one must include one more fermion than boson continuum mode in the mode regularization if one wishes to consider only TP and TAP boundary conditions. The two other boundary conditions (periodic and antiperiodic) each give in the kink sector two

fermionic solutions with  $\omega = 0$  (P) or  $\omega \sim 0$  (AP). Their coefficients yield one annihilation and one creation operator in the Dirac field expansion, and hence one corresponding fermionic degree of freedom, half localized at the kink and half localized at the boundary.

Evidently, the average of the sums with different boundary conditions is equivalent to the loss of half a fermionic degree of freedom. This half clearly is related to the half fermion charge found by Jackiw and Rebbi [13] when they considered Dirac fermions in the presence of a kink. For a Majorana fermion the half degree of freedom of the kink system was initially interpreted as one degree of freedom in the  $\bar{K}K$  system which was shared half by the kink and half by the antikink. The nonlocal character of this fermionic degree of freedom for the  $K\bar{K}$  system is still present for plain periodic boundary conditions on the kink alone: The degree of freedom is localized half at the kink (by  $\psi_1$ ) and half at the boundary (by  $\psi_2$ ). For our Majorana fermion we can interpret this fractionalization of degree of freedom as follows: Suppose one starts in the trivial vacuum with periodic boundary conditions, and one starts to rotate the right-hand half of the constant background field  $\phi$  by a Goldstone-Wilczek chiral rotation [12]. This rotation produces a current and changes the trivial vacuum to the kink vacuum. The periodic boundary condition in the kink sector stops the current, and half a degree of freedom is accumulated at the boundary. In the analysis of Jackiw and Rebbi [13] the boundary was moved to infinity and thus they only found half a charge around the kink.

The results summarized here lead to a well-defined procedure for applying mode regularization to a system in which the boundary conditions naturally change between one sector and another: In the difference of sums, require that the terms of highest energy are matched in such a way that there is no contribution to the quantum energy shift linearly divergent with the maximum energy  $\Lambda$ . Having thus matched the sums ‘at the top’ one may count down to the bottom, and compare the number of modes in each sum. For fixed boundary conditions, this procedure is equivalent to the usual mode-counting prescription. However, when the boundary conditions change, as from P+AP to TP+TAP, then the number of modes can change. (For the P, AP cases in the trivial sector and the TP case in the kink sector there are equal numbers of nonzero modes, but for the TAP case in the kink sector there is one less zero mode.) Because the number goes down by 1 for TAP but zero for TP, the average loss is  $1/2$ . Thus, one important conclusion from our work is that there is a natural generalization of mode regularization to the case when boundary conditions are not fixed. This may be useful in other contexts.

The nonlocality of one fermionic degree of freedom clearly must be an essential feature of a theory where there is only one unpaired fermion state localized at a soliton. For the case of locally invisible boundary conditions both in the trivial sector (P + AP) and in the kink sector (TP + TAP), the half charge is even more ethereal. It simply evaporates under the change in boundary conditions, as a consequence of the chiral anomaly.

The nonlocality is surprising because it appears to violate the principle of cluster decomposition. However, Majorana fermion charge is not an observable, and all vacuum expectation values for observable fields still obey the principle. Of course, if the fermion field carried an observable charge, such as fermion charge for a Dirac fermion, then the half would become a localized eigenvalue, as in the case analyzed by Jackiw and Rebbi. Thus, the un-

adorned Majorana fermion interacting with the kink is a kind of “square root” of the Dirac fermion, still manifesting the Jackiw-Rebbi half charge, but in a delocalized form, whose precise specification depends on what might otherwise have seemed an arbitrary choice of boundary conditions.

The combined thrust of all the recent works is to show that new aspects continue to appear in understanding a prototype soliton, the kink in 1+1 dimensions, with a strong hint that the methods sharpened in this theoretical laboratory are likely to have broader application.

**Acknowledgements:** We thank N. Graham, N. Manton, A. Rebhan, M. Roček, J. Schonfeld, A. Sen, M. Shifman, W. Siegel, M. Stephanov and A. Vainshtein for discussions. These results were presented at a meeting celebrating 30 years of supersymmetry in Minnesota. A new paper by Losev, Shifman and Vainshtein, [17], has discussed the multiplet structure in  $N = 1$  supersymmetric models. Their result seem to be complementary and compatible with ours.

## References

- [1] R.Rajaraman, *Solitons and instantons*, Elsevier, (1996).
- [2] A.Rebhan and P. van Nieuwenhuizen, *No saturation of the quantum Bogomolnyi bound by two-dimensional  $N=1$  supersymmetric solitons*, Nucl. Phys. **B508**, 449 (1997).
- [3] H. Nastase, M. Stephanov, P. van Nieuwenhuizen and A. Rebhan, *Topological boundary conditions, the BPS bound, and elimination of ambiguities in the quantum mass of solitons*, Nucl.Phys. **B542**, 471 (1999) , hep-th/9802074
- [4] A. Litvintsev and P. van Nieuwenhuizen, *Once more on the BPS bound for the susy kink*, hep-th/0010051
- [5] N.Graham and R.L.Jaffe, *Energy, Central Charge, and the BPS Bound for 1+1 Dimensional Supersymmetric Solitons*, Nucl. Phys. **B544**, 432 (1999), hep-th/9808140v3; N. Graham and R.L. Jaffe, *Fermionic one loop corrections to soliton energies in (1+1)-dimensions*, Nucl. Phys.**B549**, 516 (1999), hep-th/9901023; N. Graham and R.L. Jaffe, *Unambiguous One-Loop Quantum Energies of 1+1 Dimensional Bosonic Field Configurations*, Phys. Lett. **B435**, 145 (1998), hep-th/9805150; E. Farhi, N. Graham, R.L. Jaffe, and H. Weigel *Heavy Fermion Stabilization of Solitons in 1+1 Dimensions*, hep-th/0003144
- [6] M. Shifman, A. Vainshtein and M. Voloshin, *Anomaly and Quantum Corrections to Solitons in Two-Dimensional Theories with Minimal Supersymmetry*, Phys. Rev. D **59**, 45016 (1999), hep-th/9810068v2
- [7] G.V. Dunne, *Derivative Expansion and Soliton Masses*, hep-th/9907208
- [8] J.F. Schonfeld, *Soliton Masses in Supersymmetric Theories*, Nucl. Phys. **B161**, 125 (1979).
- [9] E. Witten and D. Olive, *Supersymmetry Algebras That Include Topological Charges* Phys. Lett. **B78**, 97 (1978).

- [10] N.S. Manton and T.M. Samols, *Sphalerons on a circle*, Phys. Lett. **B207**, 179 (1988).
- [11] G.A. Korn and T.M. Korn, *Mathematical handbook*, McGraw-Hill, (1968); E.T. Whittaker and G.N. Watson, *A course of modern analysis*, Camb. Univ. Press, chapter XXII (1962).
- [12] J. Goldstone and F. Wilczek, *Fractional Quantum Numbers on Solitons*, Phys. Rev. Lett. **47**, 986 (1981).
- [13] R.Jackiw and C.Rebbi, *Solitons with Fermion Number 1/2*, Phys. Rev. D **13**, 3398 (1976).
- [14] A. Ritz, M. Shifman, A. Vainshtein and M. Voloshin, *Marginal Stability and the Metamorphosis of BPS States*, hep-th/0006028
- [15] D. Binosi, M. Shifman and T. ter Veldhuis, *Leaving the BPS Bound: Tunneling of Classically Saturated Solitons*, hep-th/0006026
- [16] A. S. Goldhaber, A. Litvintsev and P. van Nieuwenhuizen, to be published
- [17] A. Losev, M. Shifman and A. Vainshtein, *Single State Supermultiplet in 1+1 Dimensions*, hep-th/0011027